

# On the Linearization of the Painlevé III-VI Equations and Reductions of the Three-Wave Resonant System

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June 12, 2007

## Abstract

We extend similarity reductions of the coupled (2+1)-dimensional three-wave resonant interaction system to its Lax pair. Thus we obtain new  $3 \times 3$  matrix Fuchs–Garnier pairs for the third, fourth, and fifth Painlevé equations, together with the previously known Fuchs–Garnier pair for the sixth Painlevé equation. These Fuchs–Garnier pairs have an important feature: they are linear with respect to the spectral parameter. Therefore we can apply the Laplace transform to study these pairs. In this way we found reductions of all pairs to the standard  $2 \times 2$  matrix Fuchs–Garnier pairs obtained by M. Jimbo and T. Miwa. As an application of the  $3 \times 3$  matrix pairs, we found an integral auto-transformation for the standard Fuchs–Garnier pair for the fifth Painlevé equation. It generates an Okamoto-like Bäcklund transformation for the fifth Painlevé equation. Another application is an integral transformation relating two different  $2 \times 2$  matrix Fuchs–Garnier pairs for the third Painlevé equation.

**2000 Mathematics Subject Classification:** 34M55, 33E17, 33E30.

**PACS 2006:** 02.30.Ik, 02.30.Gp, 02.30. Hq.

**Key words:** Three wave resonant interaction system, Painlevé equations, Lax pair, Bäcklund transformation, Laplace transform, isomonodromy deformations.

**Running title:** On Linearization of the Painlevé III-VI Equations

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# 1 Introduction

The Painlevé equations are six classical nonlinear second-order ordinary differential equations. They have been the subject of intensive investigation in the last three decades, primarily due to the fact that they appear in connection with a wide range of physical problems, including soliton systems, quantum gravity, string theory and random matrix theory. In this paper we will concentrate on the third, fourth, fifth and sixth Painlevé equations, the canonical forms of which are, respectively

$$P_3 : \quad \frac{d^2 y}{dt^2} = \frac{1}{y} \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{1}{t} (\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}, \quad (1.1)$$

$$P_4 : \quad \frac{d^2 y}{dt^2} = \frac{1}{2y} \left( \frac{dy}{dt} \right)^2 + \frac{3}{2} y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}, \quad (1.2)$$

$$P_5 : \quad \frac{d^2 y}{dt^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} + \frac{(y-1)^2}{t^2} \left( \alpha y + \frac{\beta}{y} \right) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \quad (1.3)$$

$$P_6 : \quad \frac{d^2 y}{dt^2} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left( \frac{dy}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t} \right) \frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left( \alpha + \frac{\beta t}{y^2} + \frac{\gamma(t-1)}{(y-1)^2} + \frac{\delta t(t-1)}{(y-t)^2} \right), \quad (1.4)$$

where  $\alpha, \beta, \gamma$ , and  $\delta$  are arbitrary complex parameters, see [19].

In 1888 L. Fuchs found that if the second order ODE,

$$\frac{d^2 \psi}{dx^2} = p(x) \psi, \quad (1.5)$$

where  $p(x)$  is a rational function, has a monodromy group independent of the position of singular points,  $x = t_i, i = 1, \dots, n$  (*isomonodromy deformation*), then the function  $\psi$  satisfies one more auxiliary equation,

$$\frac{\partial \psi}{\partial t_i} = A_i(x) \psi + B_i(x) \frac{\partial \psi}{\partial x}, \quad (1.6)$$

where  $A_i$  and  $B_i$  are some rational functions of  $x$ . In 1905, R. Fuchs reported that in the simplest nontrivial particular case when the equation is of the Fuchsian type with four singular points located at  $0, 1, t_1 = t, \infty$  and the fifth singular point is an apparent singularity, then its location,  $y(t)$ , is governed by  $P_6$  (see extended later article by R. Fuchs [14]). A few years later R. Garnier [15] considered the general case of the Fuchsian equation (1.5) of the second order and derived the generalization of  $P_6$  which is now known as the Garnier system. In the same paper he also found the pairs (1.5), (1.6) for the other Painlevé equations. In the latter case, equation (1.5) is non-Fuchsian. In honor of this contribution, we call the pairs that define isomonodromy deformations of linear ODEs (of arbitrary order) with rational coefficients, Fuchs–Garnier pairs.

The Fuchs–Garnier pairs associated with each Painlevé equation play a very important role in the theory and applications of the Painlevé equations. Nowadays, as a result of the intensive studies of the Painlevé equations, many different Fuchs–Garnier pairs have been derived [3, 4, 12, 18, 20, 22, 24, 26, 27, 30, 32, 33, 41]. The methods and ideas used in these derivations vary widely. As a result, most of the Painlevé equations possess a few different Fuchs–Garnier pairs whose equivalence is not yet established. Such Fuchs–Garnier pairs very often contain matrix differential equations. Thus, different Fuchs–Garnier pairs for the same Painlevé equation can have different matrix dimension, different analytic structure and even, if the two first features are the same, they can still have different *parametrization* of the matrix elements by the Painlevé functions. Our general belief is

that all different Fuchs–Garnier pairs associated with the same Painlevé equation should be related by some explicit transformations.

These transformations are interesting not only from the purely theoretical point of view, but also from a practical one. For example, for the scalar and  $2 \times 2$  matrix equations the corresponding analytic and asymptotic theories are much simpler and better developed than those for the multidimensional cases, therefore it might be useful to transport results obtained for the scalar and  $2 \times 2$  matrix Fuchs–Garnier pairs to the multidimensional case. Also, in applications, such as in geometry, the solutions of the Fuchs–Garnier pairs often have a very definite interpretation, e.g., as the functions defining embeddings of some surfaces. Therefore, explicit relations between different Fuchs–Garnier pairs, even with the same matrix dimensions, might lead to interesting insights in geometry and mathematical physics.

It is clear from the definition given above that the role of the two equations in each Fuchs–Garnier pair is not symmetric. There is a “defining” equation, namely equation (1.5) and the “deformation” equation, namely equation (1.6). The independent variable of the defining equation is called a *spectral* variable (parameter); we denote it by  $x$  or  $\lambda$ . The coefficients of both equations in Fuchs–Garnier pairs are rational functions of this variable. When the defining equation is given, the deformation equation can be derived from the isomonodromy condition. Therefore, sometimes for brevity, to present the Fuchs–Garnier pair, we write only one defining equation.

Together with the original scalar Fuchs–Garnier pairs, the  $2 \times 2$  matrix versions first presented by M. Jimbo and T. Miwa [20] play an important role in the study of the Painlevé equations. The defining equation

$$\frac{dY}{dx} = A^n(x; t)Y, \quad (1.7)$$

has the following particular forms for the Painlevé equations  $P_n$  listed above:

$$A^3(x; t) = \frac{A_0^3(t)}{x^2} + \frac{A_1^3(t)}{x} + A_2^3(t), \quad (1.8a)$$

$$A^4(x; t) = \frac{A_0^4(t)}{x} + A_1^4(t) + xA_2^4(t), \quad (1.8b)$$

$$A^5(x; t) = \frac{A_0^5(t)}{x} + \frac{A_1^5(t)}{x-1} + A_2^5(t), \quad (1.8c)$$

$$A^6(x; t) = \frac{A_0^6(t)}{x} + \frac{A_1^6(t)}{x-1} + \frac{A_t^6(t)}{x-t} \quad (1.8d)$$

The matrices  $A_i^k(t)$  are independent of the spectral parameter and are parameterized by the solutions of the corresponding Painlevé equations (see Appendix C in [20]). To distinguish other  $2 \times 2$  matrix Fuchs–Garnier pairs that are known for the same Painlevé equations, we call these pairs Fuchs–Garnier pairs in the *Jimbo–Miwa parametrization*. For convenience of the reader, we present the Fuchs–Garnier pair for  $P_5$  in Jimbo–Miwa parametrization in Appendix A.

In his studies of the Painlevé equations, Okamoto pointed out that the Painlevé equations have subgroups of symmetries isomorphic to some affine Weyl groups, [35]–[37]. Using this fact, he constructed nonlinear representations of these groups as birational canonical transformations of the Hamiltonian systems associated with the Painlevé equations. As we explain in Appendix A on the example of  $P_5$ , there is a problem with finding the linear representation of these affine Weyl groups in the space of solutions of the Fuchs–Garnier pairs in the Jimbo–Miwa parametrization. Since the latter pairs proved to be a highly effective and convenient tool for the complete description of global asymptotic properties of all solutions of the Painlevé equations and in various applications, there is a motivation to complete the theory of these Fuchs–Garnier pairs with the representation of the affine Weyl symmetries.

The goal of this paper is to create useful tools for finding transformations of solutions of Fuchs–Garnier pairs that can answer the questions raised above. Our main stimulus in this work was a recent understanding of some of the questions raised above for the case of  $P_6$  in the work by M. Mazzocco [29], D. Novikov [34] and G. Filipuk [11]. The latter two works explain that the linear representation of one nontrivial, from the isomonodromy point of view, case of Okamoto’s affine Weyl symmetries for  $P_6$  is given by the Euler integral auto-transform for the Fuchs–Garnier pair in the Jimbo–Miwa parametrization. The work by Mazzocco explains that the “dual” Fuchs–Garnier pair for  $P_6$  found by J. Harnad [18] can be mapped to the Fuchs–Garnier pair in Jimbo–Miwa parametrization by the Laplace transform<sup>1</sup>. Moreover, the linear representation of the Okamoto transformation for  $P_6$  is just a multiplication of the solution by the scalar factor  $\lambda^\alpha$  for a suitable choice of the parameter  $\alpha$ . Since multiplication by  $\lambda^\alpha$  is conjugate by the Laplace transform to the Euler transformation the result of the works [34] and [11] follows immediately. So, the  $3 \times 3$  Fuchs–Garnier pair by Harnad serves as a useful auxiliary object in this study with the Laplace transform as the main instrument.

Our objective is to extend the ideas related with the Laplace transform to the other Painlevé equations. For this purpose, we have to find proper analogues of Harnad’s Fuchs–Garnier pair for the other Painlevé equations. The adjective “proper” here means that the pairs should be  $3 \times 3$  matrix equations and we should be able to apply to them the Laplace transform. The latter condition suggests that at least the defining member of the Fuchs–Garnier pair, the ODE with respect to the spectral parameter,  $\lambda$ , should have linear coefficients in  $\lambda$ , i.e.,

$$(\lambda B_1(t) + B_2(t)) \frac{d\Phi}{d\lambda} = (\lambda B_3(t) + B_4(t)) \Phi. \quad (1.9)$$

We note that in the work by J. Harnad mentioned above, the Fuchs–Garnier pair with “spectral equation” (1.9) where  $B_2(t) = 0$ ,  $\det B_1 \neq 0$  was found for  $P_6$ . Because of that result, it is easy to understand that Fuchs–Garnier pairs with the “defining equation” (1.9) in  $3 \times 3$  matrices should exist for all Painlevé equations. Actually, M. Noumi and Y. Yamada [32, 33] found such a pair for the symmetric version of  $P_4$ . The latter pair was further studied by A. Sen, A. Hone, and P.A. Clarkson, however, in these studies, the Laplace transform was not applied and the relation with the Jimbo–Miwa Fuchs–Garnier pair (1.7), (1.8b) was not yet realized. In this paper, we report the Fuchs–Garnier pairs of the type (1.9) for  $P_3 - P_6$ : the pairs for  $P_3$ ,  $P_4$ , and  $P_5$  are new, the pair for  $P_6$  coincides with the known one [18, 29]. We note that our Fuchs–Garnier pair for  $P_4$  has a different singularity structure comparing to the one by Noumi–Yamada [33]. However, as we show, there is an invertible integral transformation linking together these pairs. We also establish the relation of the new pairs for  $P_3 - P_5$  and the Noumi–Yamada pair to the Jimbo–Miwa Fuchs–Garnier pairs (1.7), (1.8a)–(1.8c), together with the known result for  $P_6$ .

Let us remark that the phrase “linearization of the Painlevé equations” is widely understood to mean an association with some Fuchs–Garnier pair. In this paper, we extend this phrase to a “secondary” linearization, i.e., association of the Painlevé equations with Fuchs–Garnier pairs of the form (1.9). Looking ahead, we note that secondary linearization<sup>2</sup> is possible for any so-called higher-order Painlevé equations, however that is already the subject of another story.

The general form (1.9) for the defining equation of the Fuchs–Garnier pairs has matrix dimension three and, in the general case, contains too many variables for linear representations of the Painlevé equations. Instead of analyzing the general case as our starting

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<sup>1</sup> The fact that defining the equation of Harnad’s Fuchs–Garnier pair is related with the Jimbo–Miwa one via the Laplace transform was observed by Dubrovin [10]. In this connection, the work by W. Balser, W. Jurkat, and D. Lutz [2] should be mentioned also.

<sup>2</sup>In matrices with the dimension higher than 3, of course.

point, we chose another and faster way to find the the proper Fuchs–Garnier pairs, which is based on the following observations.

Recently, there appeared two independent works by R. Conte, A. M. Grundland, and M. Musette [8] and S. Kakei, T. Kikuchi [21] where the authors obtain Harnad’s Fuchs–Garnier pair for  $P_6$  by using an extension of the similarity reduction [25] for the three-wave resonant interaction (3WRI) system in  $(1+1)$  (one spatial and one time) dimensions, to the corresponding Lax pair. The Lax pair for this system was given in terms of two commuting first order differential operators in  $3 \times 3$  matrices by V. E. Zakharov and S. V. Manakov [42]. The authors of [8] and [21] were able to get the Fuchs–Garnier pair already studied by Harnad and Mazzocco and used their parametrization to get explicit formulae for the solutions of 3WRI system in terms of  $P_6$  with the complete set of the coefficients.

L. Martina and P. Winternitz in [28] obtained all classical similarity reductions for  $(2+1)$  3WRI system:

$$\frac{\partial u_j}{\partial x_j} = i u_m^* u_n^*, \quad \frac{\partial u_j^*}{\partial x_j} = -i u_m u_n, \quad i^2 = -1, \quad (1.10)$$

where  $(j, m, n)$  denotes any cyclic permutation of  $(1, 2, 3)$ ,  $u_j, u_j^*$  are the complex amplitudes of the wave packets, and star denotes complex conjugation. This system is also integrable and it possesses, of course, a much richer group of symmetries and corresponding similarity reductions than that in  $(1+1)$  dimensions. In particular, Martina and Winternitz found reductions to the  $P_6, P_5, P_4$ , and  $P_3$  equations with the complete set of the coefficients. It is important to note that *the reductions to  $P_3, P_4$ , and  $P_5$  cannot be restricted to the  $(1+1)$  case of 3WRI system.*

For each reduction Martina and Winternitz used group theoretical methods to reduce the system of three complex PDEs (1.10) to a system of three complex ODEs of the first order. In the latter system, they separated real and imaginary part to arrive at a system of six real ODEs of the first order. They showed that for all similarity reductions three of the six ODEs can be converted to one ODE of the third order which possesses the Painlevé property while the rest three can be solved in quadratures in terms of the solution of the third order equation. Such third order equations can always be integrated once to give a quite complicated ODE quadratic with respect to the second derivative, a so-called SD equation (second order second degree ODE). The latter ODEs was integrated by Bureau *et al* [5, 7] in terms of the solutions of the Painlevé equations mentioned above<sup>3</sup>. The similarity solutions obtained in this way in most cases are not explicitly written in terms of the canonical Painlevé functions, because in the papers [5, 7] solutions of the SD equations are not always presented in simple form in terms of the canonical functions. So in this study no any techniques related with the Lax pairs were involved.

We note that solutions of the 3WRI system (1.10) are not analytic, and therefore working with that system we cannot achieve our goal – to get Fuchs–Garnier pairs for the general case of the Painlevé equations – without any artificial restrictions. So, we have to consider an analytic extension of the 3WRI system which we call also the coupled 3WRI system. We do it in the standard way, namely, we forget that the upper script  $*$  means complex conjugation in the six equations in (1.10) and we consider  $u_j$  and  $u_j^*$  as independent complex functions. The coupling procedure spoils neither its integrability, so that formally the same Lax pair serves for the coupled version of 3WRI system, nor the Martina–Winternitz similarity reductions.

To construct secondary linearized Fuchs–Garnier pairs for the Painlevé equations (1.1)–(1.4), we have to find for each Martina–Winternitz similarity reduction its extension to the Lax pair for the coupled 3WRI system. At this stage we arrive at  $3 \times 3$  matrix Fuchs–Garnier pairs for a system of ODEs defining similarity solutions of the coupled 3WRI system. A substantial question here is how to introduce the spectral parameter; since

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<sup>3</sup>For recent advanced results see the paper by C. M. Cosgrove and G. Scoufis [9].

originally the Lax pair for the 3WRI system does not contain any spectral parameters, this is the major difference with the situation for  $(1+1)$  integrable systems, where the Lax pairs already possess the spectral parameter. While there are many papers in the literature concerning extensions of the similarity reductions of  $(1+1)$  integrable systems to their Lax pairs, this methodology is well known since the work of H. Flashka and A. C. Newell [12], we do not know such works for  $(2+1)$  integrable systems. We note that in our case the reduction cannot be done successively:  $(2+1) \rightarrow (1+1) \rightarrow (1+0)$ . We found that for all similarity reductions it is possible to introduce the spectral parameter such that the defining equations of the resulting Fuchs–Garnier pairs gain the form (1.9).

After the  $3 \times 3$  matrix Fuchs–Garnier pairs are obtained we use Laplace and/or gauge transformations to map these pairs to the  $2 \times 2$  Fuchs–Garnier pairs in the Jimbo–Miwa parametrization. Comparing parameterizations between the one that comes from the coupled 3WRI system and the Jimbo–Miwa parametrization we obtain explicit formulae for the similarity solutions in terms of the canonical Painlevé functions. This comparison also allows us to parameterize the  $3 \times 3$  Fuchs–Garnier pairs in terms of the canonical Painlevé functions, i.e., to obtain secondary linearized Fuchs–Garnier pairs for the Painlevé equations.

We also consider parametrization of the similarity solutions for the physical case of our coupled 3WRI system, i.e., the original 3WRI system. At this stage we also arrive at SD functions but in this approach they have a lucid sense as the Hamiltonians ( $\tau$ -functions) for the Painlevé equations.

The paper consists of six Sections and one Appendix. Section 1 is the Introduction. In Section 2 we recall the Lax pair for the 3WRI system. Sections 3–6 represent the main body of the paper: each one is devoted to the corresponding Painlevé equation beginning with  $P_6$  and finishing with  $P_3$ . The Sections are divided into Subsections which represent logical steps of the derivation indicated above: similarity reductions from Martina and Winternitz, extensions of the reduction to the Lax pair, reductions via the Laplace transform to the Fuchs–Garnier pairs in the Jimbo–Miwa parameterizations, parameterizations of similarity solutions by the Painlevé functions. Sections 4 and 6 have also Subsections with the alternate reductions of the  $3 \times 3$  matrix Fuchs–Garnier pairs to the  $2 \times 2$  ones. Section 4 contains also one more extra subsection with the derivation of the Okamoto transformation. Appendix A is devoted to the spectral interpretation of the Bäcklund transformations for  $P_5$ . In particular, we define the Okamoto transformation and, at the very end, present the alternate parametrization of isomonodromy deformations for equation (1.7), (1.8c).

The main results obtained in this work are as follows:

1. We introduced a notion of secondary linearization for the Painlevé equations as the Fuchs–Garnier pairs with the defining equation (1.9) in  $3 \times 3$  matrices. We found these pairs for  $P_3$ – $P_6$ . Three pairs for  $P_3$ ,  $P_4$ , and  $P_5$  are new. The pair for  $P_6$  coincides with the Harnad one [18], see Subsections 3.2, 4.2, 5.2, 6.2. We prove that the pair for  $P_4$  found in this paper is equivalent to the pair for the symmetric form of this equation by Noumi and Yamada [32, 33], but the singularity structure of our one is different, see Subsection 5.4;
2. We found a relation of all secondary linearized pairs to the Fuchs–Garnier pairs in the Jimbo–Miwa parametrization. This is a new result only for the pairs for  $P_3$ ,  $P_4$ , and  $P_5$ : see Subsections 4.2, 5.2, 6.2;
3. For  $P_5$  we found an explicit linear representation for the nontrivial Okamoto affine Weyl symmetry. It is given as an integral auto-transform of the solution  $Y$  of the Jimbo–Miwa Fuchs–Garnier pair. The mechanism of its derivation is different from that for the analogous result for  $P_6$ , see Subsection 4.5;
4. For the  $2 \times 2$  Fuchs–Garnier pair for  $P_5$  with the defining equation (1.7), (1.8c) we found a simpler and more natural parametrization, which we call the “true” Jimbo–Miwa parametrization, see Appendix A;

5. For both cases of  $P_3$  (the complete and degenerate) two  $2 \times 2$  Fuchs–Garnier pairs are known see, e.g. [22, 27], we found that they are related via an integral transform, see Subsection 6.4;
6. As a byproduct of our work, for both coupled and physical cases of 3WRI system and for all similarity reductions to the Painlevé equations explicit parameterizations in terms of the canonical Painlevé functions are obtained: see Subsections 3.3, 4.3, 5.3, 6.3.

The secondary linearization also exists, of course, for the first and second Painlevé equations. They are not related with the similarity reductions of 3WRI system and corresponding results will be published separately.

We expect that this approach with the auxiliary secondary linearized Fuchs–Garnier pairs will be very fruitful for the hierarchies of the Painlevé equations.

## 2 Lax Pair for the 3WRI System

System (1.10) admits a Lax pair found by Kaup [23]. We write it here in a modified form with the spectral parameter  $k$ :

$$\begin{aligned} \frac{\partial \psi_j}{\partial x_m} - ik\kappa_m \psi_j &= -iu_n^* \psi_m \\ \frac{\partial \psi_m}{\partial x_j} - ik\kappa_j \psi_m &= iu_n \psi_j \end{aligned} \quad (2.1)$$

where  $(j, m, n)$  denotes any cyclic permutation of  $(1, 2, 3)$ ,  $\psi_j = \psi_j(x_m, k)$  are scalar functions,  $\kappa_j$  are real constants, and  $k \in \mathbb{C}$  is the spectral parameter. We note that our notation differs from Kaup's one by the factor,  $\psi_j \mapsto \psi_j \exp[ik(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)]$ , for all  $j = 1, 2, 3$ . The spectral parameter  $k$  appeared in Kaup's analysis of the scattering problem for the system (2.1) for a different class of solutions of (1.10).

System (2.1) can be written in matrix form in the following way

$$\begin{aligned} \mathcal{D}_1 \Psi &= i(kK_1 + U_1) \Psi \\ \mathcal{D}_2 \Psi &= i(kK_2 + U_2) \Psi \end{aligned} \quad (2.2)$$

where  $\Psi$  is a  $3 \times 3$  matrix-valued function, the matrix operators  $\mathcal{D}_1, \mathcal{D}_2$  and the matrices  $K_1, K_2$  and  $U_1, U_2$  are defined as follows:

$$\mathcal{D}_1 = \text{diag}[\partial_{x_2}, \partial_{x_3}, \partial_{x_1}], \quad \mathcal{D}_2 = \text{diag}[\partial_{x_3}, \partial_{x_1}, \partial_{x_2}],$$

$$\begin{aligned} K_1 &= \begin{pmatrix} \kappa_2 & 0 & 0 \\ 0 & \kappa_3 & 0 \\ 0 & 0 & \kappa_1 \end{pmatrix}, & U_1 &= \begin{pmatrix} 0 & -u_3^* & 0 \\ 0 & 0 & -u_1^* \\ -u_2^* & 0 & 0 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} \kappa_3 & 0 & 0 \\ 0 & \kappa_1 & 0 \\ 0 & 0 & \kappa_2 \end{pmatrix}, & U_2 &= \begin{pmatrix} 0 & 0 & u_2 \\ u_3 & 0 & 0 \\ 0 & u_1 & 0 \end{pmatrix}. \end{aligned}$$

We note that, when written in standard cartesian coordinates, the linear system (2.2) is equivalent to the Lax pair considered by Fokas and Ablowitz in [13].

In the following sections we will investigate the particular similarity reductions found in [28] that are linked to  $P_3$ ,  $P_4$ ,  $P_5$  and  $P_6$ , giving an explicit extension of each reduction on the Lax pair (2.2).

### 3 Similarity Reduction to the Sixth Painlevé Equation

The following similarity reduction for system (1.10) was obtained in [25] and [28]:

$$u_1 = (x_2 - x_3)^{-1+i\rho_1} v_1, \quad u_2 = (x_1 - x_3)^{-1+i\rho_2} v_2, \quad u_3 = (x_1 - x_2)^{-1+i\rho_3} v_3, \quad (3.1)$$

where  $v_j = v_j(\tau)$  with

$$\tau = \frac{x_1 - x_3}{x_2 - x_3}, \quad (3.2)$$

and  $\rho_1, \rho_2, \rho_3$  are real constants such that

$$\rho_1 + \rho_2 + \rho_3 = 0. \quad (3.3)$$

Under this reduction, system (1.10) reduces to the following system of ODEs:

$$\begin{aligned} \tau^{1+i\rho_2}(\tau-1)^{1+i\rho_3} v_1' &= i v_2^* v_3^* \\ \tau^{i\rho_2}(\tau-1)^{1+i\rho_3} v_2' &= -i v_3^* v_1^* \\ \tau^{1+i\rho_2}(\tau-1)^{i\rho_3} v_3' &= i v_1^* v_2^*, \end{aligned} \quad (3.4)$$

where prime denotes differentiation with respect to  $\tau$ .

It is mentioned in the Introduction that the above system was integrated directly in [28] in terms of the general solution of an SD equation which, in turn, is solvable in terms of the sixth Painlevé function. We will show at the end of Subsection 3.3 that the similarity solutions can be written in a (relatively) simple way in terms of the canonical  $P_6$  functions, so that in this case the SD equation is an intermediate object that makes the formulae cumbersome. For the other similarity reductions, SD functions are actually needed.

The one-dimensional restriction of the similarity reduction (3.1) was used in the recent works [8] and [21] to obtain a  $3 \times 3$  Fuchs–Garnier pair for  $P_6$  from the  $(1+1)$ -dimensional scattering Lax pair. In the remainder of this section we will rederive this result from the  $(2+1)$ -dimensional perspective.

**Remark 3.1.** However before we generalize this similarity reduction to the coupled case of the 3WRI system. One adds to (3.1) and (3.4) the formally conjugated equations

$$\begin{aligned} u_1^* &= (x_2 - x_3)^{-1-i\rho_1} v_1^*, \quad u_2^* = (x_1 - x_3)^{-1-i\rho_2} v_2^*, \quad u_3^* = (x_1 - x_2)^{-1-i\rho_3} v_3^*, \\ \tau^{1-i\rho_2}(\tau-1)^{1-i\rho_3} v_1^{*'} &= -i v_2 v_3, \\ \tau^{-i\rho_2}(\tau-1)^{1-i\rho_3} v_2^{*'} &= i v_3 v_1, \\ \tau^{1-i\rho_2}(\tau-1)^{-i\rho_3} v_3^{*'} &= -i v_1 v_2. \end{aligned} \quad (3.5)$$

Note that in the coupled case  $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$  satisfy the same relation (3.3) and the functions  $v_j$  and  $v_j^*$  are not assumed to be complex conjugates. In the most part of this Section we deal with the coupled 3WRI system and turn back to the physical case at the end of Subsection 3.3.

#### 3.1 The $3 \times 3$ Fuchs–Garnier Pair

To compute the reduced Lax pair we introduce the spectral parameter  $\lambda$  in the following way

$$\lambda = (x_2 - x_3)k.$$

Writing  $\Psi(x_j, k) = R(x_j) \tilde{\Phi}(\tau, \lambda)$ , where  $R(x_j)$  is given by

$$R(x_1, x_2, x_3) = \text{diag}((x_2 - x_3)^{i\theta_{23}}, (x_2 - x_3)^{i\theta_{31}}, (x_2 - x_3)^{i\theta_{12}}),$$



and

$$\theta_{12} - \theta_{31} = \rho_1, \quad \theta_{23} - \theta_{12} = \rho_2, \quad \theta_{31} - \theta_{23} = \rho_3,$$

we find that Lax pair (2.2) can be rewritten as follows:

$$\begin{aligned} C_1 \tilde{\Phi}_\tau + \lambda D_1 \tilde{\Phi}_\lambda &= i(\lambda K_1 + V_1) \tilde{\Phi} \\ C_2 \tilde{\Phi}_\tau + \lambda D_2 \tilde{\Phi}_\lambda &= i(\lambda K_2 + V_2) \tilde{\Phi}, \end{aligned}$$

where the matrices  $C_j, D_j, K_j, V_j$  are given by

$$\begin{aligned} C_1 &= \text{diag}(-\tau, \tau-1, 1), & C_2 &= \text{diag}(\tau-1, 1, -\tau), \\ D_1 &= \text{diag}(1, -1, 0), & D_2 &= \text{diag}(-1, 0, 1), \\ K_1 &= \text{diag}(\kappa_2, \kappa_3, \kappa_1), & K_2 &= \text{diag}(\kappa_3, \kappa_1, \kappa_2), \\ V_1 &= \begin{pmatrix} -\theta_{23} & -(\tau-1)^{-1-i\rho_3} v_3^* & 0 \\ 0 & \theta_{31} & -v_1^* \\ -\tau^{-1-i\rho_2} v_2^* & 0 & 0 \end{pmatrix}, \\ V_2 &= \begin{pmatrix} \theta_{23} & 0 & \tau^{-1+i\rho_2} v_2 \\ (\tau-1)^{-1+i\rho_3} v_3 & 0 & 0 \\ 0 & v_1 & -\theta_{12} \end{pmatrix}. \end{aligned}$$

After rearranging the above system, we find

$$\tilde{\Phi}_\lambda = \left( Q^{(1)} + \frac{Q^{(0)}}{\lambda} \right) \tilde{\Phi} \quad (3.7a)$$

$$\tilde{\Phi}_\tau = \left( \lambda P^{(1)} + P^{(0)} \right) \tilde{\Phi}, \quad (3.7b)$$

where the matrices  $Q^{(1)}, P^{(1)}, Q^{(0)}, P^{(0)}$  are given by

$$Q^{(1)} = i \text{diag}(-(\tau-1)\kappa_2 - \tau\kappa_3, (\tau-1)\kappa_1 - \kappa_3, \tau\kappa_1 + \kappa_2), \quad (3.8a)$$

$$P^{(1)} = i \text{diag}(-\kappa_2 - \kappa_3, \kappa_1, \kappa_1), \quad (3.8b)$$

and

$$Q^{(0)} = i \begin{pmatrix} -\theta_{23} & (\tau-1)^{-i\rho_3} v_3^* & -\tau^{i\rho_2} v_2 \\ (\tau-1)^{i\rho_3} v_3 & -\theta_{31} & v_1^* \\ -\tau^{-i\rho_2} v_2^* & v_1 & -\theta_{12} \end{pmatrix}, \quad (3.8c)$$

$$P^{(0)} = i \begin{pmatrix} 0 & (\tau-1)^{-1-i\rho_3} v_3^* & -\tau^{-1+i\rho_2} v_2 \\ (\tau-1)^{-1+i\rho_3} v_3 & 0 & 0 \\ -\tau^{-1-i\rho_2} v_2^* & 0 & 0 \end{pmatrix}. \quad (3.8d)$$

In order to integrate the reduced system (3.4), (3.5) in terms of  $P_6$  we compare the Fuchs–Garnier representation (3.7) with the  $3 \times 3$  Fuchs–Garnier representation for  $P_6$  obtained in [18] and [29].

**Remark 3.2.** As noted earlier, the spectral parameter  $k$  has been introduced formally into Lax pair (2.2). We made use of this fact in extending the similarity reduction (3.1) to obtain a similarity reduction for the associated Lax pair. Here we would like to illustrate that, although introduction of the auxiliary spectral parameter  $k$  is not absolutely necessary, in this particular case it is an important ingredient of our construction of the Fuchs–Garnier pair.

An alternate construction is also possible in which the spectral variable  $\lambda$  is introduced without any dependence on  $k$ . Writing  $\tilde{\Psi} = \Psi \exp[ik(\kappa_1 x_1 + \kappa_2 x_2 + \kappa_3 x_3)]$  in (2.2), we then introduce the spectral variable as  $\lambda = (x_2 - x_3)$  and follow the procedure described

above. In fact, we can just put  $\kappa_1 = \kappa_2 = \kappa_3 = 0$  and  $k = 1$  in (3.7)–(3.8). As a result we arrive at the following  $3 \times 3$  matrix representation for the reduced system (3.4), (3.5):

$$\lambda \frac{d\tilde{\Phi}}{d\lambda} = Q^{(0)}\tilde{\Phi}, \quad \frac{d\tilde{\Phi}}{d\tau} = P^{(0)}\tilde{\Phi},$$

where  $Q^{(0)}, P^{(0)}$  are given in (3.8). This is also a  $3 \times 3$  Fuchs–Garnier pair for the similarity solutions; one can still get first integrals for system (3.4), (3.5) as eigenvalues of  $Q^{(0)}$ , however all further information about the solutions is hidden in a normalization of this system rather than encoded in the monodromy structure. Therefore this system is ineffective for further studying of the similarity solutions.

### 3.2 Reduction of the $3 \times 3$ Fuchs–Garnier Pair to the $2 \times 2$ Pair in Jimbo–Miwa Form

Now we simplify the notation and rewrite Fuchs–Garnier pair (3.7) in the following form:

$$\Phi_\lambda = \left( B_1^6 + \frac{B_0^6 - I}{\lambda} \right) \Phi \tag{3.9a}$$

$$\Phi_t = \left( \lambda M_1^6 + M_0^6 \right) \Phi, \tag{3.9b}$$

where the matrices  $B_1^6, M_1^6, B_0^6, M_0^6$  are given by

$$B_1^6 = \text{diag}(t, 1, 0), \quad M_1^6 = \text{diag}(1, 0, 0),$$

and

$$B_0^6 = \begin{pmatrix} -\theta_2 & \tilde{w}_3 & w_2 \\ w_3 & -\theta_3 & \tilde{w}_1 \\ \tilde{w}_2 & w_1 & -\theta_1 \end{pmatrix}, \quad M_0^6 = \begin{pmatrix} 0 & (t-1)^{-1}\tilde{w}_3 & t^{-1}w_2 \\ (t-1)^{-1}w_3 & 0 & 0 \\ t^{-1}\tilde{w}_2 & 0 & 0 \end{pmatrix},$$

where  $\{w_j, \tilde{w}_j\}$  are functions of  $t$  and  $\theta_1, \theta_2, \theta_3$  are arbitrary constants. Following [29] we assume that 0 is one of eigenvalues of the matrix  $B_0^6(t)$ . Note that this condition is a normalization of system (3.9) rather than a restriction. If we denote the eigenvalues of the matrix  $B_0^6(t)$  as  $\mu_1, \mu_2$ , and  $\mu_3$ , then we can write:

$$\mu_1 = \frac{1}{2} \left( -\sum_{j=1}^3 \theta_j + \theta_\infty \right), \quad \mu_2 = \frac{1}{2} \left( -\sum_{j=1}^3 \theta_j - \theta_\infty \right), \quad \mu_3 = 0,$$

where  $\theta_\infty$  is an arbitrary constant. We note that system (3.9a) coincides exactly with the system given in [29] if we make the gauge transformation  $\Phi \mapsto J\hat{\Phi}$  where  $J$  is the constant matrix

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We omit writing the compatibility condition for pair (3.9), which coincides with system (3.4), (3.5) rewritten in terms of variables  $w_j, \tilde{w}_j$  (see equations (3.17)), because we do not use it. Instead, following [29], we briefly outline how this pair can be mapped to the  $2 \times 2$  Fuchs–Garnier pair for  $P_6$  given by Jimbo and Miwa, which is defined by equation (1.8d). As a result we obtain a parametrization of the 3WRI system in terms of the solutions of  $P_6$ . We present this parametrization in the next section.

We introduce the function  $\tilde{Y}(x, t)$  via the generalized Laplace transform

$$\Phi(\lambda, t) = \int_C e^{\lambda x} \tilde{Y}(x, t) dx. \tag{3.10}$$

Before substituting (3.10) into equations (3.9), it is convenient to rewrite equation (3.9a) as follows,

$$\lambda\Phi_\lambda = (\lambda B_1^6(t) + B_0^6(t) - \mathbf{I})\Phi,$$

Assuming that the contour  $C$  in (3.10) can be chosen to eliminate any remainder terms that arise from integration-by-parts, we find

$$(B_1^6(t) - xI) \frac{d\tilde{Y}}{dx} = B_0^6(t)\tilde{Y}, \quad \frac{d\tilde{Y}}{dt} = -M_1^6 \frac{d\tilde{Y}}{dx} + M_0^6 \tilde{Y}.$$

Substituting the first equation obtained above into the second one we obtain:

$$\frac{d\tilde{Y}}{dx} = (B_1^6 - xI)^{-1} B_0^6 \tilde{Y}, \quad \frac{d\tilde{Y}}{dt} = (-M_1^6 (B_1^6 - xI)^{-1} B_0^6 + M_0^6) \tilde{Y}, \quad (3.11)$$

Since one of the eigenvalues of  $B_0^6$ , which are integrals of motion, is 0, we can choose the Jordan form of  $B_0^6$  such that all the elements of its last column are zeroes. Denote such Jordan form as  $\hat{B}_0^6$ . If  $B_0^6$  is diagonalizable, then  $\hat{B}_0^6 = \text{diag}[\mu_1, \mu_2, 0]$ . Define  $G_0$ ,  $\det G_0 = 1$ , as follows  $G_0^{-1} B_0^6(t_0) G_0 = \hat{B}_0^6$  at some point  $t_0 \neq 0, 1, \infty$ . Now, define matrix  $G$ , as a solution of the equation  $\frac{dG}{dt} = M_0^6 G$  satisfying the initial data  $G(t_0) = G_0$ . It is easy to observe that the compatibility conditions for Fuchs–Garnier pair (3.9) implies that  $G^{-1} B_0^6(t) G = \hat{B}_0^6$  holds for all  $t$ . We make the gauge transformation  $\hat{Y} = G\tilde{Y}$  in system (3.11), to find the following Fuchsian system for  $\hat{Y}$ :

$$\frac{d\hat{Y}}{dx} = \left( \frac{\hat{A}_0^6(t)}{x} + \frac{\hat{A}_t^6(t)}{x-t} + \frac{\hat{A}_1^6(t)}{x-1} \right) \hat{Y}, \quad \frac{d\hat{Y}}{dt} = -\frac{\hat{A}_t^6(t)}{x-t} \hat{Y}, \quad (3.12)$$

where the  $3 \times 3$  matrices  $\hat{A}_j^6$  all have the form

$$\hat{A}_j^6 = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

Since the third column of each  $\hat{A}_j^6$  is zero, the system for  $\hat{Y}$  reduces to a system for the first two components

$$\frac{dY}{dx} = \left( \frac{A_0^6(t)}{x} + \frac{A_t^6(t)}{x-t} + \frac{A_1^6(t)}{x-1} \right) Y, \quad \frac{dY}{dt} = -\frac{A_t^6(t)}{x-t} Y, \quad Y = \begin{pmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{pmatrix}, \quad (3.13)$$

and a quadrature for the third component. The eigenvalues of the matrices  $A_0^6$ ,  $A_t^6$  and  $A_1^6$  are  $(\theta_1, 0)$ ,  $(\theta_2, 0)$  and  $(\theta_3, 0)$ , respectively, see [29]. Equation (3.13) is equivalent (up to gauge transformation) to the  $2 \times 2$  Fuchs–Garnier system for  $P_6$  in the form given by Jimbo and Miwa in [20]. Now comparing with the Jimbo–Miwa parametrization of the matrix elements of (3.13) by solutions of  $P_6$ , we arrive at the parametrization for  $w_j, \tilde{w}_j$  presented in the next section.

### 3.3 Similarity Solution of 3WRI System in Terms of the Sixth Painlevé Equation

More details of the calculation explained in the previous section can be found in [29]. Here we present the final result, the parametrization of the functions  $w_j, \tilde{w}_j$  in terms of

$P_6$  together with the corresponding reduction to get solutions of the 3WRI system.

$$w_1 = f \left( \frac{(t-1)y' - \theta_1(y-1)}{2y} + \frac{\theta_3(t-1) + (\theta_\infty - 1)(y-1)}{2t} \right), \quad (3.14a)$$

$$\tilde{w}_1 = f^{-1} \left( -\frac{\theta_3 y - t y'}{2(y-1)} + \frac{\theta_1 t + (\theta_\infty - 1)y}{2(t-1)} \right), \quad (3.14b)$$

$$w_2 = \frac{g}{f} \left( -\frac{\theta_2 y + t y'}{2(y-t)} - \frac{\theta_1 + \theta_\infty y}{2(t-1)} + \frac{y(y-1)}{2(t-1)(y-t)} \right), \quad (3.14c)$$

$$\tilde{w}_2 = \frac{f}{g} \left( \frac{t(t-1)y' - \theta_1(y-t)}{2y} - \frac{\theta_2(t-1) - \theta_\infty(y-t) + y-1}{2} \right), \quad (3.14d)$$

$$w_3 = g^{-1} \left( -\frac{\theta_3(y-t) + t(t-1)y'}{2(y-1)} + \frac{\theta_2 t - \theta_\infty(y-t) + y}{2} \right), \quad (3.14e)$$

$$\tilde{w}_3 = g \left( -\frac{(t-1)y' + \theta_2(y-1)}{2(y-t)} + \frac{\theta_3 - \theta_\infty(y-1)}{2t} + \frac{y(y-1)}{2t(y-t)} \right), \quad (3.14f)$$

where the functions  $f = f(t)$  and  $g = g(t)$  are the general solutions of the following equations:

$$\frac{d}{dt} \log f = -\frac{y'}{2y(y-1)} - \frac{1 + \theta_1 - \theta_2 + \theta_3}{2t(t-1)} + \frac{\theta_1}{2(t-1)y} + \frac{\theta_3}{2t(y-1)}, \quad (3.15)$$

$$\begin{aligned} \frac{d}{dt} \log g = & \frac{y' - 1}{2(y-t)} - \frac{y'}{2(y-1)} + \frac{1 - \theta_1 + \theta_2 - \theta_3}{2t} + \theta_2 \left( \frac{1}{t-1} + \frac{1}{2(y-t)} \right) \\ & + \theta_3 \left( -\frac{1}{t(t-1)} + \frac{1}{2t(y-1)} \right), \end{aligned} \quad (3.16)$$

and  $y(t)$  is a solution of  $P_6$  with

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_1^2}{2}, \quad \gamma = \frac{\theta_3^2}{2}, \quad \delta = \frac{1 - \theta_2^2}{2},$$

In order to solve the reduced coupled 3WRI system (3.4), (3.5) in terms of  $P_6$  we put  $\kappa_1 = 0, \kappa_2 = 0, \kappa_3 = i$  in (2.2), and then compare matrix entries in the Fuchs–Garnier pair (3.7) with  $\tau = t$  with those in system (3.9). We obtain the following correspondence:

$$i v_1(t) = w_1(t), \quad -i t^{i\rho_2} v_2(t) = w_2(t), \quad i(t-1)^{i\rho_3} v_3(t) = w_3(t), \quad (3.17a)$$

$$i v_1^*(t) = \tilde{w}_1(t), \quad -i t^{-i\rho_2} v_2^*(t) = \tilde{w}_2(t), \quad i(t-1)^{-i\rho_3} v_3^*(t) = \tilde{w}_3(t), \quad (3.17b)$$

and

$$i\rho_1 = \theta_1 - \theta_3, \quad i\rho_2 = \theta_2 - \theta_1, \quad i\rho_3 = \theta_3 - \theta_2. \quad (3.17c)$$

Now we consider the physical reduction, i.e., assume that the star in (3.4) denotes the complex conjugation. First of all we have to impose the reduction on the formal monodromies:

$$\theta_1 = i\theta_{12}, \quad \theta_2 = i\theta_{23}, \quad \theta_3 = i\theta_{31},$$

where  $\theta_{ik} \in \mathbb{R}$  and  $\theta_\infty \in \mathbb{R}$ . The solution  $y(t)$  should be real for real  $t$ , and the functions  $f$  and  $g$  are as follows:

$$\begin{aligned} f(t) &= \frac{\sqrt{y} \sqrt{t}}{\sqrt{y-1} \sqrt{t-1}} \left| \frac{t}{t-1} \right|^{\frac{\theta_1 - \theta_2 + \theta_3}{2}} \exp \left( \frac{\theta_3}{2} \int_{t_0}^t \frac{dt}{t(y-1)} + \frac{\theta_1}{2} \int_{t_0}^t \frac{dt}{(t-1)y} + i c_1 \right), \\ g(t) &= \frac{\sqrt{y-t} \sqrt{t}}{\sqrt{y-1}} |t|^{\frac{\theta_2 - \theta_1 + \theta_3}{2}} |t-1|^{\theta_2 - \theta_3} \exp \left( \frac{\theta_2}{2} \int_{t_0}^t \frac{dt}{y-t} + \frac{\theta_3}{2} \int_{t_0}^t \frac{dt}{t(y-1)} + i c_2 \right), \end{aligned}$$

where the parameters  $t_0, c_1, c_2 \in \mathbb{R}$ , the parameter  $c_1 \neq 0$  only in the case if  $\theta_3 = \theta_1 = 0$ , and  $c_2 \neq 0$  if  $\theta_3 = \theta_2 = 0$ . Moreover, the solution of  $P_6$  should satisfy the following condition:  $0 < t < 1$  and  $t < y(t) < 1$ .

**Remark 3.3.** We note that the parametrization that was adopted in [29] to write system (3.9) explicitly in terms of  $y$  where  $y(t)$  is a solution of  $P_6$  is not unique. Alternate parameterizations have been identified by Boalch [3]–[4] in his studies of  $P_6$ . Since this system can be mapped to the irregular  $3 \times 3$  Lax pair of [18] and [29] via the generalized Laplace transform, see equation (3.12) above, it follows that these parameterizations are equivalent to system (3.9) up to a gauge transformation.

## 4 Similarity Reduction to the Fifth Painlevé Equation

We consider the following similarity reduction of the 3WRI system, which was obtained in [28],

$$u_1 = e^{-ix_2x_3}x_3^{\frac{i\rho}{2}}v_1, \quad u_2 = e^{ix_3x_1}x_3^{\frac{i\rho}{2}}v_2, \quad u_3 = (x_1 - x_2)^{-1+i\rho}v_3, \quad (4.1)$$

where  $v_j = v_j(\tau)$  with

$$\tau = (x_1 - x_2)x_3, \quad (4.2)$$

and  $\rho$  is a real constant. Under these assumptions system (1.10) reduces to the system of ODEs:

$$\tau^{1+i\rho}e^{i\tau}v_1' = iv_2^*v_3^*, \quad \tau^{1+i\rho}e^{i\tau}v_2' = -iv_3^*v_1^*, \quad \tau^{i\rho}e^{i\tau}v_3' = iv_1^*v_2^*. \quad (4.3)$$

where prime denotes differentiation with respect to  $\tau$ . This system was integrated in [28] in terms of an SD-function and shown to be solvable in terms of the fifth Painlevé equation (1.3).

**Remark 4.1.** It is straightforward to generalize this similarity reduction to the coupled case of the 3WRI system. One adds to (4.1) and (4.3) the formally conjugated equations

$$u_1^* = e^{ix_2x_3}x_3^{-\frac{i\rho}{2}}v_1^*, \quad u_2^* = e^{-ix_3x_1}x_3^{-\frac{i\rho}{2}}v_2^*, \quad u_3^* = (x_1 - x_2)^{-1-i\rho}v_3^*, \\ \tau^{1-i\rho}e^{-i\tau}v_1^{*'} = -iv_2v_3, \quad \tau^{1-i\rho}e^{-i\tau}v_2^{*'} = +iv_3v_1, \quad \tau^{-i\rho}e^{-i\tau}v_3^{*'} = -iv_1v_2.$$

Note that in the coupled case  $\rho \in \mathbb{C}$  and the functions  $v_j$  and  $v_j^*$  are not assumed to be complex conjugates. In the most part of this Section we deal with the coupled 3WRI system and turn back to the physical case at the end of Subsection 4.3.

### 4.1 Fuchs–Garnier Pair for the Reduced System

Following the approach outlined in the previous section we will use (4.1) to construct a  $3 \times 3$  Fuchs–Garnier pair for the reduced system (4.3). The pair is valid in the coupled case also.

Consider Lax pair (2.2). In this case we introduce the spectral parameter in a different way comparing with the previous section: instead of a scaled version of the spectral parameter  $k$ , the new spectral parameter  $\lambda$  is defined in terms of the dynamical variables, namely,

$$\lambda = (x_1 + x_2)x_3.$$

Setting  $\kappa_1 = \kappa_2 = \kappa_3 = 0$  in (2.2) one proves that the solution of the Lax pair has the following similarity form,

$$\Psi = R(x_j)\tilde{\Phi}(\tau, \lambda), \quad R(x_1, x_2, x_3) = \text{diag} \left( e^{ix_1x_3}x_3^{-i\theta_{23}}, e^{ix_2x_3}x_3^{-i\theta_{31}}, x_3^{-1-i\theta_{12}} \right),$$

where

$$\theta_{12} - \theta_{31} = -\frac{\rho}{2}, \quad \theta_{23} - \theta_{12} = -\frac{\rho}{2}, \quad \theta_{31} - \theta_{23} = \rho. \quad (4.4)$$

In terms of the new variables the Lax pair (2.2) becomes

$$\begin{aligned}\tau\tilde{\Phi}_\tau + D_1\tilde{\Phi}_\lambda &= i\left(-\frac{1}{2}(\lambda - \tau)S_2 + V_1\right)\tilde{\Phi} \\ \tau\tilde{\Phi}_\tau + D_2\tilde{\Phi}_\lambda &= i\left(-\frac{1}{2}(\lambda + \tau)S_1 + V_2\right)\tilde{\Phi},\end{aligned}$$

where the matrices  $D_j, S_j, V_j$  are given by

$$\begin{aligned}D_1 &= \text{diag}(-\tau, \lambda, \tau), & D_2 &= \text{diag}(\lambda, \tau, -\tau), \\ S_1 &= \text{diag}(1, 0, 0), & S_2 &= \text{diag}(0, 1, 0), \\ V_1 &= \begin{pmatrix} 0 & \tau^{-i\rho}e^{-i\tau}v_3^* & 0 \\ 0 & \theta_{31} & -v_1^* \\ -\tau v_2^* & 0 & 0 \end{pmatrix}, & V_2 &= \begin{pmatrix} \theta_{23} & 0 & v_2 \\ \tau^{i\rho}e^{i\tau}v_3 & 0 & 0 \\ 0 & -\tau v_1 & 0 \end{pmatrix}.\end{aligned}$$

After rearranging, the above system can be written as

$$\tilde{\Phi}_\lambda = \left(\frac{Q^{(0)}}{\lambda + \tau} + \frac{Q^{(1)}}{\lambda - \tau} + Q^{(2)}\right)\tilde{\Phi} \quad (4.5a)$$

$$\tilde{\Phi}_\tau = \left(\frac{Q^{(0)}}{\lambda + \tau} - \frac{Q^{(1)}}{\lambda - \tau} + P^{(2)}\right)\tilde{\Phi} \quad (4.5b)$$

where the matrices  $Q^{(0)}, Q^{(1)}$  and  $Q^{(2)}, P^{(2)}$  are given by

$$Q^{(0)} = i \begin{pmatrix} \theta_{23} & -\tau^{-i\rho}e^{-i\tau}v_3^* & v_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad Q^{(1)} = i \begin{pmatrix} 0 & 0 & 0 \\ -\tau^{i\rho}e^{i\tau}v_3 & \theta_{31} & -v_1^* \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$Q^{(2)} = -\frac{i}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ v_2^* & -v_1 & 0 \end{pmatrix}, \quad P^{(2)} = -\frac{i}{2} \begin{pmatrix} 1 & -2\tau^{-1-i\rho}e^{-i\tau}v_3^* & 0 \\ -2\tau^{-1+i\rho}e^{i\tau}v_3 & -1 & 0 \\ v_2^* & v_1 & 0 \end{pmatrix}.$$

## 4.2 Fuchs–Garnier Pairs for the Fifth Painlevé Equation

The goal of this section is to establish a map between the  $3 \times 3$  Fuchs–Garnier pair (4.5) and the  $2 \times 2$  Fuchs–Garnier pair for  $P_5$  found by Jimbo and Miwa [20]. It is convenient to introduce the “coupled notation” for matrix elements of the Fuchs–Garnier pair:

$$\Phi_\lambda = \left(\frac{B_1}{\lambda + t} + \frac{B_2}{\lambda - t} + \frac{1}{2}I + B_3\right)\Phi, \quad (4.6a)$$

$$\Phi_t = \left(\frac{B_1}{\lambda + t} - \frac{B_2}{\lambda - t} + M_\infty\right)\Phi, \quad (4.6b)$$

where

$$B_1 = \begin{pmatrix} \tilde{m} & \tilde{w}_3 & w_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ w_3 & m & \tilde{w}_1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{w}_2/2 & w_1/2 & -1/2 \end{pmatrix}, \quad (4.7)$$

where  $I$  is the identity matrix and

$$M_\infty = \begin{pmatrix} 1/2 & -t^{-1}\tilde{w}_3 & 0 \\ -t^{-1}w_3 & -1/2 & 0 \\ \tilde{w}_2/2 & -w_1/2 & 0 \end{pmatrix}, \quad (4.8)$$

where  $\{w_j, \tilde{w}_j\}$  are all functions of  $t$ .

Compatibility of equations (4.6a) and (4.6b) gives the following system of equations:

$$m' = 0, \quad \tilde{m}' = 0, \quad (4.9)$$

and

$$\begin{aligned} tw'_1 &= \tilde{w}_2 \tilde{w}_3, & t\tilde{w}'_1 &= -w_2 w_3, \\ tw'_2 &= -\tilde{w}_1 \tilde{w}_3, & t\tilde{w}'_2 &= w_1 w_3, \\ tw'_3 &= -[t - (m - \tilde{m})]w_3 - t\tilde{w}_1 \tilde{w}_2, & t\tilde{w}'_3 &= [t - (m - \tilde{m})]\tilde{w}_3 + tw_1 w_2, \end{aligned} \quad (4.10)$$

where the primes denote derivatives by  $t$ .

As in the previous section, we are going to apply to the Fuchs–Garnier pair the generalized Laplace transform. For this purpose we rewrite equations (4.6) in the appropriate form with the coefficients linearly depending on the spectral parameter, namely:

$$(\lambda J_0 + tJ)\Phi_\lambda = \left(\frac{1}{2}(\lambda J_0 + tJ) + B\right)\Phi, \quad (4.11a)$$

$$(\lambda J_0 + tJ)\Phi_t = ((\lambda J_0 + tJ)M + JB)\Phi, \quad (4.11b)$$

where

$$J_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.12a)$$

$$B = B_1 + B_2 + tB_3 = \begin{pmatrix} \tilde{m} & \tilde{w}_3 & w_2 \\ w_3 & m & \tilde{w}_1 \\ \frac{t}{2}\tilde{w}_2 & \frac{t}{2}w_1 & -\frac{t}{2} \end{pmatrix}, \quad (4.12b)$$

$$M = M_\infty - B_3 = \begin{pmatrix} 1/2 & -\tilde{w}_3/t & 0 \\ -w_3/t & -1/2 & 0 \\ 0 & -w_1 & 1/2 \end{pmatrix}, \quad (4.12c)$$

We define the generalized Laplace transform as follows,

$$\Phi(\lambda, t) = \int_C e^{\lambda x/2} \tilde{Y}(x, t) dx. \quad (4.13)$$

Substituting it into equations (4.11a) and (4.11b), and assuming that the contour  $C$  can be suitably chosen to eliminate any remainder terms that arise from integration-by-parts, we find

$$(x-1)J_0 \frac{d\tilde{Y}}{dx} = \left(\frac{t}{2}(x-1)J - (J_0 + B)\right)\tilde{Y}, \quad (4.14)$$

$$(J_0 + B) \frac{d\tilde{Y}}{dt} = \left(\frac{x-1}{2}(J + JB + tJM - tJ_0MJ) + J_0M(J_0 + B) - \frac{d}{dt}B\right)\tilde{Y}, \quad (4.15)$$

where in the derivation of equation (4.15) we used equation (4.14). The third row of equation (4.14) reads:

$$x\tilde{Y}_3 = \tilde{w}_2\tilde{Y}_1 + w_1\tilde{Y}_2. \quad (4.16)$$

Using this relation to eliminate  $\tilde{Y}_3$  from (4.14) and (4.15) we obtain the following  $2 \times 2$  system:

$$\frac{dY}{dx} = \left(\frac{t}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} w_2\tilde{w}_2 & w_1w_2 \\ \tilde{w}_1\tilde{w}_2 & w_1\tilde{w}_1 \end{pmatrix} - \frac{1}{x-1} \begin{pmatrix} w_2\tilde{w}_2 + \tilde{m} + 1 & \tilde{w}_3 + w_1w_2 \\ w_3 + \tilde{w}_1\tilde{w}_2 & w_1\tilde{w}_1 + m + 1 \end{pmatrix}\right)Y, \quad (4.17)$$

$$\frac{dY}{dt} = \left(\frac{x}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{t} \begin{pmatrix} 0 & -\tilde{w}_3 \\ -w_3 & 0 \end{pmatrix}\right)Y, \quad Y = \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix}. \quad (4.18)$$

The Fuchs–Garnier pair (4.17), (4.18) coincides (up to a simple gauge transformation) with the Fuchs–Garnier pair for  $P_5$  by Jimbo–Miwa (see [20], equations (C.38), (C.39)).

### 4.3 Parametrization of Solutions in Terms of $P_5$

In order to parameterize the general solution of system (4.10) by the (general) solution of  $P_5$  we compare the parametrization  $2 \times 2$  Fuchs–Garnier representations for this system obtained above (4.17), (4.18) with the one by Jimbo and Miwa [20].

First of all we notice that system (4.10) admits first integrals

$$\begin{aligned} m = \text{const}, \quad \tilde{m} - m = \theta_\infty, \quad w_1 \tilde{w}_1 + w_2 \tilde{w}_2 = \theta_0 \\ w_1 w_2 w_3 + \tilde{w}_1 \tilde{w}_2 \tilde{w}_3 + w_3 \tilde{w}_3 + \frac{\theta_\infty}{2}(w_2 \tilde{w}_2 - w_1 \tilde{w}_1) = \frac{\theta_1^2 - \theta_0^2 - \theta_\infty^2}{4}, \end{aligned} \quad (4.19)$$

where  $\theta_0, \theta_1, \theta_\infty$  are arbitrary constants, which have a sense of formal monodromies of the normalized solution  $Y$  of system (4.17), (4.18). “Normalized” here means that we make a gauge transformation of  $Y$  which puts all matrices in (4.17) into the traceless form. The notation of the formal monodromies coincides with those from the Jimbo–Miwa work [20]. The first integral  $m$  cannot be expressed via the monodromies because the normalized version of equation (4.17) depends only on the difference  $\tilde{m} - m$ , rather than on  $m$  and  $\tilde{m}$  separately.

Motivated by the parametrization used by Jimbo–Miwa we define the functions

$$y(t) = \frac{w_2 \tilde{w}_2 (\tilde{w}_3 + w_1 w_2)}{(w_2 \tilde{w}_2 - (\theta_0 + \theta_1 - \theta_\infty)/2) w_1 w_2}, \quad z(t) = w_2 \tilde{w}_2 - \theta_0, \quad u(t) = -\frac{w_1}{\tilde{w}_2}. \quad (4.20)$$

It follows from system (4.10) and parametrization (4.19) that  $y, z, u$  satisfy the following system of nonlinear ODEs:

$$ty' = ty - 2z(y-1)^2 - \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) (y-1)^2 + (\theta_0 + \theta_1)(y-1), \quad (4.21a)$$

$$tz' = yz \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - \frac{1}{y}(z + \theta_0) \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right), \quad (4.21b)$$

$$t(\log u)' = -2z - \theta_0 + y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) + \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right). \quad (4.21c)$$

System (4.21) coincides with the system (C.40) in [20]. Eliminating  $z$  from the first equation and substituting it into the second one we find that  $y(t)$  satisfies the general  $P_5$  equation (1.3) with the coefficients:

$$\alpha = \frac{1}{2} \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right)^2, \quad \beta = -\frac{1}{2} \left( \frac{\theta_0 - \theta_1 - \theta_\infty}{2} \right)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \quad \delta = -1/2. \quad (4.22)$$

Now we find the converse formulae, namely, the functions  $\{w_j(t), \tilde{w}_j(t)\}$  in terms of  $y(t)$  and  $z(t)$  and  $u(t)$ . Using (4.20) we obtain the following representations for  $\{w_j(t), \tilde{w}_j(t)\}$ :

$$\begin{aligned} w_1 &= -fz(z + \theta_0), \quad w_2 = \frac{1}{gz}, \\ \tilde{w}_1 &= \frac{1}{f(z + \theta_0)}, \quad \tilde{w}_2 = gz(z + \theta_0), \\ w_3 &= \frac{g}{f} \left( \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) - z \right), \\ \tilde{w}_3 &= -\frac{f}{g} \left( y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - (z + \theta_0) \right), \end{aligned} \quad (4.23)$$

where instead of one function  $u(t)$  we are forced to introduce two functions  $f(t)$  and  $g(t)$ , such that  $u(t) = f(t)/g(t)$ . The additional function appears as a result of an extra



“gauge freedom” in the  $3 \times 3$  Fuchs–Garnier pair compared to the  $2 \times 2$  one. System (4.10) implies, that the functions  $f(t)$  and  $g(t)$  satisfy the following equations:

$$\begin{aligned} t(\log f)' &= -\frac{tz'}{z+\theta_0} - \frac{tz'}{2z} + \frac{1}{2} \left( y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - (z + \theta_0) \right) \\ &\quad + \frac{z + \theta_0}{2z} \left( \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) - z \right), \\ t(\log g)' &= -\frac{tz'}{z} - \frac{tz'}{2(z+\theta_0)} - \frac{z}{2(z+\theta_0)} \left( y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - (z + \theta_0) \right) \\ &\quad - \frac{1}{2} \left( \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) - z \right). \end{aligned} \quad (4.24)$$

These expressions can be simplified by introducing the function  $\sigma(t)$ , following the work of Jimbo and Miwa in [20]. In our notation, we define  $\sigma(t)$  as

$$\sigma(t) = w_3 \tilde{w}_3 + t w_1 \tilde{w}_1 + \frac{(\theta_0 + \theta_\infty)^2 - \theta_1^2}{4}. \quad (4.25)$$

Then, using the fourth identity in (4.19), we find

$$t(\log f)' = -\frac{tz'}{z+\theta_0} - \frac{tz'}{2z} + \frac{1}{2z} \left( \sigma + (t + \theta_\infty) \sigma' \right), \quad (4.26a)$$

$$t(\log g)' = -\frac{tz'}{z} - \frac{tz'}{2(z+\theta_0)} - \frac{1}{2(z+\theta_0)} \left( \sigma + (t + \theta_\infty) \sigma' \right). \quad (4.26b)$$

We note that the function  $\sigma(t)$  satisfies the following two important equations:

$$\frac{d\sigma}{dt} = -z(t), \quad (4.27)$$

which can be proved by the differentiation of equation (4.25), and

$$\begin{aligned} t^2 \left( \frac{d^2 \sigma}{dt^2} \right)^2 &= \left( \sigma - (\theta_\infty + 2\theta_0 + t) \frac{d\sigma}{dt} + 2 \left( \frac{d\sigma}{dt} \right)^2 \right)^2 \\ &\quad - 4 \frac{d\sigma}{dt} \left( \frac{d\sigma}{dt} - \theta_0 \right) \left( \frac{d\sigma}{dt} - \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) \left( \frac{d\sigma}{dt} - \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right). \end{aligned} \quad (4.28)$$

Equation (4.28) can be verified in the following way. Substituting the parametrization (4.23) into equation (4.25), we express the  $\sigma$ -function in terms of  $y$  and  $z$ . We then couple the resulting expression with equation (4.21b) and use (4.27) to eliminate  $z$ . Then, summing up and subtracting these equations one finds the two equations:

$$\begin{aligned} t\sigma'' + (\sigma - (\theta_\infty + 2\theta_0 + t)\sigma' + 2\sigma'^2) &= 2 \frac{\sigma' - \theta_0}{y} \left( \sigma' - \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right), \\ t\sigma'' - (\sigma - (\theta_\infty + 2\theta_0 + t)\sigma' + 2\sigma'^2) &= -2\sigma'y \left( \sigma' - \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right), \end{aligned} \quad (4.29)$$

The compatibility condition of these equations is equivalent to (4.28). Thus  $\sigma(t)$  is the so-called  $SD$ -function.

We are now ready to solve the reduced 3WRI system (4.3) in terms of  $P_5$ . The Fuchs–Garnier pairs (4.5) and (4.6) are related by the change of variables  $\lambda \mapsto i\lambda, \tau \mapsto i\tau$ . By comparing matrix entries between (4.5) and (4.6) we get the following correspondence

$$v_1(\tau) = -w_1(t), \quad v_2(\tau) = -iw_2(t), \quad v_3(\tau) = i\tau^{-i\rho} e^{-i\tau} w_3(t), \quad (4.30a)$$

$$v_1^*(\tau) = i\tilde{w}_1(t), \quad v_2^*(\tau) = \tilde{w}_2(t), \quad v_3^*(\tau) = i\tau^{i\rho} e^{i\tau} \tilde{w}_3(t), \quad (4.30b)$$

$$i\theta_{23} = \tilde{m}, \quad i\theta_{31} = m, \quad \rho = \theta_{31} - \theta_{23} = i(\tilde{m} - m) = i\theta_\infty. \quad (4.30c)$$

where the functions  $\{w_j, \tilde{w}_j\}$  are given in terms of  $y$  and  $z$  by equations (4.23) and (4.24). These formulae define the general similarity solution for the coupled case of the 3WRI system. We note that as follows from equations (4.27) and (4.29) the solution of the coupled system can be presented in terms one function  $\sigma(t)$ .

To give the general similarity solution in the physical case of the 3WRI system, i.e. where we prove that the functions  $v_j(\tau)$  and  $v_j^*(\tau)$ ,  $j = 1, 2, 3$ , are complex conjugates for real  $\tau$ , it is necessary to present the solution solely in terms of the function  $\sigma(t)$ . To achieve this we impose the following conditions on the parameters

$$t, \theta_0, \theta_1, \theta_\infty \in i\mathbb{R}. \quad (4.31)$$

Introducing the notation  $\tilde{\sigma}(\tau) = \sigma(t)$ , where  $\tau = it$ , we note that  $\tilde{\sigma}$  satisfies an ODE analogous to (4.28) which, by condition (4.31), will have real coefficients. It follows that we can take the general solution  $\tilde{\sigma}(\tau)$  of this equation to be real. After making the change of variables  $t = -i\tau$ ,  $\theta_\infty = -i\rho$ ,  $\theta_0 = -i\rho_0$ ,  $\theta_1 = -i\rho_1$  with  $\tau, \rho, \rho_0, \rho_1 \in \mathbb{R}$ , we define the functions  $z(t)$  and  $y(t)$  by equations (4.27) and any one of (4.29), respectively. Then, we use (4.26) to obtain the following expressions for the functions  $f$  and  $g$ :

$$\begin{aligned} \frac{1}{f(z + \theta_0)} &= z^{1/2} \exp \left( -i \int_{\tau_0}^{\tau} \frac{\tilde{\sigma} - (\tau + \rho)\tilde{\sigma}'}{2\tilde{\sigma}'} d\tau + if_0 \right), \\ \frac{1}{gz} &= (z + \theta_0)^{1/2} \exp \left( i \int_{\tau_0}^{\tau} \frac{\tilde{\sigma} - (\tau + \rho)}{2(\tilde{\sigma}' + \rho_0)} d\tau + ig_0 \right), \end{aligned} \quad (4.32)$$

Notice that the function  $z(t) = -i\tilde{\sigma}'(\tau)$  is a pure imaginary function of  $\tau$ . Assume that  $e^{i\pi/2}z > 0$  and  $e^{-i\pi/2}(z + \theta_0) > 0$ . It is straightforward now to observe from equations (4.23) and (4.30) that the functions  $v_j$  and  $v_j^*$  for  $j = 1, 2$  are indeed conjugates under conditions (4.31). To see that the same is true for the functions  $v_3$  and  $v_3^*$ , one has to employ additionally equations (4.29).

#### 4.4 Alternate Reduction to the $2 \times 2$ Fuchs–Garnier Pair for $P_5$

In this section we present an alternate reduction of the  $3 \times 3$  system (4.6) to the Jimbo–Miwa version of the Fuchs–Garnier system for  $P_5$  [20], by making use of suitable gauge transformations rather than the generalized Laplace transform. The key observation is that the parameter  $m$  in matrix  $B$  (see equation (4.12b)) can be chosen such that its determinant vanishes for all values of  $t$ . Indeed,

$$\det B = w_1 w_2 w_3 + \tilde{w}_1 \tilde{w}_2 \tilde{w}_3 + w_3 \tilde{w}_3 - \tilde{m} w_1 \tilde{w}_1 - m w_2 \tilde{w}_2 - m \tilde{m},$$

is the first integral of system (4.10) by virtue of equations (4.19). Moreover, by putting

$$m = -\frac{\theta_0 + \theta_1 + \theta_\infty}{2}, \quad (4.33)$$

one finds that  $\det B$  coincides with the difference of the l.h.s. and r.h.s. of the last integral in (4.19) and thus vanishes for all  $t^4$ . It follows that  $B$  has eigenvalues  $(\mu_1, \mu_2, 0)$ , where  $\mu_k = \mu_k(t)$ ,  $k = 1, 2$ . Moreover, on the **general** solutions<sup>5</sup> of system (4.10) all eigenvalues are pairwise different, thus there exists an invertible matrix  $G = G(t)$  such that

$$G^{-1}BG = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv \tilde{B}. \quad (4.34)$$

---

<sup>4</sup>Parameter  $\theta_1$  is defined in (4.19) up to the sign, therefore one can change  $\theta_1 \rightarrow -\theta_1$  in the definition of  $m$  in (4.33).

<sup>5</sup>We assume that the general situation holds in this section.

In Section 4.5 we give explicit expressions for the eigenvalues  $\mu_1(t), \mu_2(t)$  and the diagonalizing matrix  $G(t)$  in terms of the functions  $\{w_j, \tilde{w}_j\}$ ,  $j = 1, 2, 3$ .

We now make the gauge transformation  $\Phi = G\tilde{\mathcal{Y}}$  in system (4.11a), (4.11b) to obtain

$$\frac{d\tilde{\mathcal{Y}}}{d\lambda} = \left( \frac{1}{2}I + \tilde{A}_3 + \frac{\tilde{A}_2}{\lambda - t} + \frac{\tilde{A}_1}{\lambda + t} \right) \tilde{\mathcal{Y}}, \quad \frac{d\tilde{\mathcal{Y}}}{dt} = \left( \tilde{M}_\infty - \frac{\tilde{A}_2}{\lambda - t} + \frac{\tilde{A}_1}{\lambda + t} \right) \tilde{\mathcal{Y}}, \quad (4.35)$$

where

$$\tilde{A}_k = G^{-1}B_kG = G^{-1}I_kBG = G^{-1}I_kG\tilde{B}, \quad (4.36)$$

$$\tilde{M}_\infty = G^{-1} \left( M_\infty G - \frac{d}{dt}G \right), \quad (4.37)$$

the matrices  $B_k$ ,  $M_\infty$ , and  $\tilde{B}$  are defined by equations (4.7), (4.8), and (4.34), respectively, and, for each  $k = 1, 2, 3$ , the matrix  $I_k$  has only one nonzero element, which is the  $k$ -th element on the diagonal, more precisely,  $I_k \equiv \{\delta_{ik}\delta_{kj}\}_{i,j=1}^{i,j=3}$ , where  $\delta_{nm}$  is Kroneker's delta.

Note that from the first equation (4.36) follows that  $\text{rank } \tilde{A}_k = 1$ . Let us prove that the first two elements in the third columns of the matrices  $\tilde{M}_\infty$  and  $\tilde{A}_k$  are zeroes:

$$\tilde{M}_\infty[1, 3] = \tilde{M}_\infty[2, 3] = \tilde{A}_k[1, 3] = \tilde{A}_k[2, 3] = A_k[3, 3] = 0, \quad k = 1, 2, 3. \quad (4.38)$$

For the matrix elements of  $\tilde{A}_k$  equations (4.38) are an immediate consequence of the second equation (4.36). For the matrix  $\tilde{M}_\infty$  it follows from the compatibility condition of system (4.35),  $\tilde{A}'_k = [\tilde{M}_\infty, \tilde{A}_k]$ , where the brackets denote the matrix commutator, the corresponding structure of the matrices  $\tilde{A}_k$ , see equations (4.38), and the fact that  $\text{rank } B = 2$  according to the assumption in footnote 5.

$$\tilde{M}_\infty = \begin{pmatrix} \boxed{\hat{M}_\infty} & 0 \\ & 0 \\ * & * & * \end{pmatrix}, \quad \tilde{A}_k = \begin{pmatrix} \boxed{\hat{A}_k} & 0 \\ & 0 \\ * & * & 0 \end{pmatrix}, \quad k = 1, 2, 3. \quad (4.39)$$

Note that  $\tilde{M}_\infty[3, 3] \neq 0$ .

Thus the structure of the matrices  $\tilde{M}_\infty$  and  $\tilde{A}_k$  (4.39) implies that system (4.35) can be reduced to a system in  $2 \times 2$  matrices for the first two components of  $\tilde{\mathcal{Y}}$ , while the third component can be found in terms of the first two via a quadrature. The reduced  $2 \times 2$  system looks the same as system (4.35) the only difference is that tildes are changed by hats.

To proceed let us notice that the eigenvalues of the matrices  $\tilde{A}_1$ ,  $\tilde{A}_2$ , and  $\tilde{A}_3$ , equal  $\{\tilde{m}, 0, 0\}$ ,  $\{m, 0, 0\}$ , and  $\{-1/2, 0, 0\}$ , correspondingly; it follows from the definition of  $\tilde{A}_k$  and  $B_k$  see the first equation in (4.36) and equations (4.7), respectively. The structure of matrices  $\tilde{A}_k$  in (4.39) implies that the matrices  $\hat{A}_1$ ,  $\hat{A}_2$ , and  $\hat{A}_3$ , have the following eigenvalues:  $\{\tilde{m}, 0\}$ ,  $\{m, 0\}$ , and  $\{-1/2, 0\}$ , respectively. Therefore, the matrix  $1/2I + \hat{A}_3$  has eigenvalues  $\{1/2, 0\}$  and there exists an invertible matrix  $H$ , such that,

$$H^{-1}(\frac{1}{2}I + \hat{A}_2)H = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}, \quad (4.40)$$

Thus, the function

$$\mathcal{Y} \equiv H^{-1}\hat{\mathcal{Y}} = H^{-1} \begin{pmatrix} \tilde{\mathcal{Y}}_{11} & \tilde{\mathcal{Y}}_{12} \\ \tilde{\mathcal{Y}}_{21} & \tilde{\mathcal{Y}}_{22} \end{pmatrix},$$

where  $\hat{\mathcal{Y}}$  is the corresponding main minor of any solution of system (4.35), solves the Fuchs–Garnier pair:

$$\frac{d\mathcal{Y}}{d\lambda} = \left( \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} + \frac{\mathcal{A}_2}{\lambda - t} + \frac{\mathcal{A}_1}{\lambda + t} \right) \mathcal{Y}, \quad \frac{d\mathcal{Y}}{dt} = \left( \mathcal{D}_\infty - \frac{\mathcal{A}_2}{\lambda - t} + \frac{\mathcal{A}_1}{\lambda + t} \right) \mathcal{Y}, \quad (4.41)$$

where

$$\mathcal{A}_k = H^{-1} \hat{A}_k H \quad \text{for } k = 1, 2 \quad \text{and} \quad \mathcal{D}_\infty = H^{-1} \left( \hat{M}_\infty H - \frac{d}{dt} H \right). \quad (4.42)$$

We remark that  $\mathcal{D}_\infty$  is a diagonal matrix for any  $H$  in (4.40). Indeed, the compatibility condition for system (4.41) implies,

$$[\mathcal{D}_\infty, \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix}] = 0.$$

Finally, to put the Fuchs–Garnier pair (4.41), into the Jimbo–Miwa form for  $P_5$  [20], we have to make the following transformation:

$$\mathcal{Y}(\lambda, t) = e^{\lambda/4} x^{\tilde{m}/2} (x-1)^{m/2} D(t) Y(x, t), \quad \lambda = 2tx - t,$$

where  $x$  is the new spectral parameter and  $D$  is a diagonal matrix depending only on  $t$  defined as follows,

$$D^{-1} \frac{d}{dt} D = \mathcal{D}_\infty + \frac{1}{t} \text{diag}(\mathcal{A}_1 + \mathcal{A}_2) - \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4.43)$$

The function  $Y$  solves the system

$$\frac{dY}{dx} = \left( \frac{t}{2} \sigma_3 + \frac{A_0}{x} + \frac{A_1}{x-1} \right) Y, \quad \frac{dY}{dt} = \left( \frac{x}{2} \sigma_3 + \frac{1}{t} \text{offdiag}(A_0 + A_1) \right) Y, \quad (4.44)$$

where the notation  $\text{offdiag}(\cdot)$  means the off-diagonal part of the corresponding matrix, i.e., the matrix where the diagonal elements are substituted by zeroes,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_0 = D^{-1} \mathcal{A}_1 D - \frac{\tilde{m}}{2} I, \quad A_1 = D^{-1} \mathcal{A}_2 D - \frac{m}{2} I.$$

The matrices  $A_k$  obey the following relations:

$$\text{tr } A_0 = \text{tr } A_1 = 0, \quad \det A_0 = -\frac{\tilde{m}^2}{4}, \quad \det A_1 = -\frac{m^2}{4}, \quad \text{diag}(A_0 + A_1) = -\frac{\theta_1 - \theta_0}{2} \sigma_3.$$

The results for the traces and determinants of  $A_0$  and  $A_1$  can be deduced from the corresponding results for the matrices  $\mathcal{A}_1$  and  $\mathcal{A}_2$ :

$$\text{tr } \mathcal{A}_1 = \tilde{m}, \quad \text{tr } \mathcal{A}_2 = m, \quad \det \mathcal{A}_1 = \det \mathcal{A}_2 = 0,$$

which are proved above. To prove the formula for the diagonal part of  $A_0 + A_1$ , we note that actually the following more general formula is valid,

$$A_0 + A_1 = D^{-1} H^{-1} \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} H D + \begin{pmatrix} 0 & 0 \\ 0 & t/2 \end{pmatrix} - \frac{m + \tilde{m}}{2} I. \quad (4.45)$$

To prove identity (4.45) one has to start with the formula  $B_1 + B_2 = B - B_3$  and follow the construction presented in this section. Then use formula (4.47) for  $H$  given in Subsection 4.5, to prove that the diagonal part of equation (4.45) equals  $(\theta_0 + \frac{m+\tilde{m}}{2})\sigma_3$ . Finally recall the choice of  $m$  in (4.33) and the equation for  $\tilde{m}$  in (4.19).

By the way, since the trace of l.h.s. of (4.45) equals 0, we find that  $\mu_1 + \mu_2 = m + \tilde{m} - t/2$ , which is consistent with equation (4.46) of Subsection 4.5.

## 4.5 An Okamoto-type Bäcklund Transformation for $P_5$

In Subsections 4.2 and 4.4 we found two different reductions of the  $3 \times 3$  Fuchs–Garnier pair (4.6) to the  $2 \times 2$  Fuchs–Garnier pair of the Jimbo–Miwa type, namely, (4.17), (4.18) and (4.44). In this Subsection we present some details of the calculations related with the reduction scheme of the previous Subsection. Using them the interested reader can follow the same scheme as in Subsection 4.3 to get an alternate parametrization of the similarity reduction (4.1), (4.2) of the 3WRI system in terms of solutions of  $P_5$ . We, however, proceed in a different way: we find a specific Okamoto-type Bäcklund transformation for  $P_5$  (see Appendix A equation (A.21)) together with the generating integral transformation for solutions of the Fuchs–Garnier pair.

So we begin with the explicit formulae for the objects introduced in the previous Subsection: The diagonalizing matrix  $G(t)$  in (4.34) is taken as

$$G = \begin{pmatrix} m w_2 - \tilde{w}_1 \tilde{w}_3 - w_2 \mu_1 & m w_2 - \tilde{w}_1 \tilde{w}_3 - w_2 \mu_2 & m w_2 - \tilde{w}_1 \tilde{w}_3 \\ \tilde{m} \tilde{w}_1 - w_2 w_3 - \tilde{w}_1 \mu_1 & \tilde{m} \tilde{w}_1 - w_2 w_3 - \tilde{w}_1 \mu_2 & \tilde{m} \tilde{w}_1 - w_2 w_3 \\ -\frac{t}{2}(\mu_2 + \theta_0 + \frac{t}{2}) & -\frac{t}{2}(\mu_1 + \theta_0 + \frac{t}{2}) & -(\mu_1 + \frac{t}{2})(\mu_2 + \frac{t}{2}) - \frac{t}{2}\theta_0 \end{pmatrix}$$

where  $\mu_1$  and  $\mu_2$  are solutions of the following quadratic equation,

$$\mu^2 - (m + \tilde{m} - \frac{t}{2})\mu - (w_3 \tilde{w}_3 + \frac{t}{2}(m + \tilde{m} + \theta_0) - m\tilde{m}) = 0. \quad (4.46)$$

In the general situation all three eigenvalues  $\{\mu_1, \mu_2, 0\}$  are different,

$$\det G = ((m - \tilde{m})w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3)(\mu_2 - \mu_1)\mu_1 \mu_2 \neq 0.$$

The diagonalizing matrix  $H(t)$  in (4.40) is taken as

$$H = \begin{pmatrix} 1 & -\frac{\mu_2}{\mu_1} \\ -\frac{(2\mu_2 + t + 2\theta_0)\mu_1}{(2\mu_1 + t + 2\theta_0)\mu_2} & 1 \end{pmatrix}, \quad \det H = \frac{2(\mu_1 - \mu_2)}{2\mu_1 + t + 2\theta_0}. \quad (4.47)$$

An important auxiliary object is the diagonal matrix  $D$ , the logarithmic derivative of which is defined in equation (4.43). A nontrivial ingredient of the formula in (4.43) is the diagonal matrix  $\mathcal{D}_\infty$  defined in the second equation in (4.42). Using MAPLE code and following the algorithm of Subsection 4.4 one confirms that  $\mathcal{D}_\infty$  is, indeed, the diagonal matrix. This calculation at the same time gives extremely complicated expressions for the diagonal elements:  $\mathcal{D}_\infty[1, 1]$ ,  $\mathcal{D}_\infty[2, 2]$ . We were not able to find a concise expression for them. At the same time it is not complicated to find an expression for the logarithmic derivative of the ratio  $D_{11}/D_{22}$  of the diagonal elements of  $D$ , or, equivalently, the difference  $\mathcal{D}_\infty[1, 1] - \mathcal{D}_\infty[2, 2]$ , see below.

Using the formulae for  $G(t)$ ,  $H(t)$  and  $D(t)$  given above we obtain the following expressions for the matrices  $A_0$ ,  $A_1$  in (4.44):

$$A_1 = \begin{pmatrix} \frac{(w_2 w_3 - \tilde{m} \tilde{w}_1)((\theta_0 + \tilde{m})w_2 + \tilde{w}_1 \tilde{w}_3)}{(m - \tilde{m})w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3} - \frac{\tilde{m}}{2} \\ \frac{D_{11} \mu_1 (\tilde{m}(\theta_0 + m) \tilde{w}_1 - \theta_0 w_2 w_3 - \tilde{w}_1 w_3 \tilde{w}_3)((\theta_0 + \tilde{m})w_2 + \tilde{w}_1 \tilde{w}_3)}{D_{22} \mu_2 (\mu_2 + \theta_1)((m - \tilde{m})w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3)} \\ \frac{D_{22} \mu_2 (\mu_2 + \theta_1) w_2 (w_2 w_3 - \tilde{m} \tilde{w}_1)}{D_{11} \mu_1 ((m - \tilde{m})w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3)} \\ \frac{w_2 (\tilde{m}(\theta_0 + m) \tilde{w}_1 - \theta_0 w_2 w_3 - \tilde{w}_1 w_3 \tilde{w}_3)}{(m - \tilde{m})w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3} - \frac{\tilde{m}}{2} \end{pmatrix},$$

$$\begin{aligned}
A_2 = & \begin{pmatrix} \frac{(m w_2 - \tilde{w}_1 \tilde{w}_3)((\theta_0 + m) \tilde{w}_1 + w_2 w_3)}{(m - \tilde{m}) w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3} - \frac{m}{2} \\ -\frac{D_{11} \mu_1 (m(\theta_0 + \tilde{m}) w_2 - \theta_0 \tilde{w}_1 \tilde{w}_3 - w_2 w_3 \tilde{w}_3)((\theta_0 + m) \tilde{w}_1 + w_2 w_3)}{D_{22} \mu_2 (\mu_2 + \theta_1)((m - \tilde{m}) w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3)} \\ \frac{D_{22} \mu_2 (\mu_2 + \theta_1) \tilde{w}_1 (m w_2 - \tilde{w}_1 \tilde{w}_3)}{D_{11} \mu_1 ((m - \tilde{m}) w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3)} \\ -\frac{\tilde{w}_1 (m(\theta_0 + \tilde{m}) w_2 - \theta_0 \tilde{w}_1 \tilde{w}_3 - w_2 w_3 \tilde{w}_3)}{(m - \tilde{m}) w_2 \tilde{w}_1 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3} - \frac{m}{2} \end{pmatrix}, \\
A_1 + A_2 = & \begin{pmatrix} \frac{\theta_0 - \theta_1}{2} & -\frac{D_{22} \mu_2 (2 \mu_1 + 2 \theta_0 + t)}{2 D_{11} \mu_1} \\ \frac{D_{11} \mu_1 (2 \mu_2 + 2 \theta_0 + t)}{2 D_{22} \mu_2} & -\frac{\theta_0 - \theta_1}{2} \end{pmatrix}. \quad (4.48)
\end{aligned}$$

Equation (4.48) is obtained as a sum of the matrices  $A_0$  and  $A_1$  presented above. However, we used identities for  $\mu_1$  and  $\mu_2$  following from equation (4.46) to simplify the off diagonal elements. The same formula can be obtained in a different way: from equation (4.45) with matrix  $H$  in (4.47).

Now we compare the Jimbo-Miwa parametrization of system (4.44) ([20] equation (C.38))<sup>6</sup> with the one obtained above. To differentiate from the solution of  $P_5$  that already appeared in Subsection 4.3 we adopt the hat notation: the solution of  $P_5$ ,  $\hat{y} = \hat{y}(t)$ , associated functions  $\hat{z} = \hat{z}(t)$  and  $\hat{u} = \hat{u}(t)$ , see system (4.21), and the corresponding monodromies  $\hat{\theta}_0, \hat{\theta}_1, \hat{\theta}_\infty$ , which we obtain in this section. Thus we arrive at the following equations for the formal monodromies:

$$\hat{\theta}_0 = -\tilde{m} = \frac{\theta_0 + \theta_1 - \theta_\infty}{2}, \quad \hat{\theta}_1 = m = -\frac{\theta_0 + \theta_1 + \theta_\infty}{2}, \quad \hat{\theta}_\infty = \theta_1 - \theta_0, \quad (4.49)$$

The choice of the signs for  $\hat{\theta}_0$  and  $\hat{\theta}_1$  in equations (4.49) are in our hands (see Appendix A). After we fixed the signs we obtain equations for the  $P_5$  functions:

$$\begin{aligned}
\hat{z} = & -\frac{(\tilde{m} \tilde{w}_1 - w_2 w_3)((\theta_0 + \tilde{m}) w_2 + \tilde{w}_1 \tilde{w}_3)}{(m - \tilde{m}) \tilde{w}_1 w_2 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3}, \\
\hat{z} + \hat{\theta}_0 = & -\frac{w_2 (\tilde{m}(\theta_0 + m) \tilde{w}_1 - \theta_0 w_2 w_3 - \tilde{w}_1 w_3 \tilde{w}_3)}{(m - \tilde{m}) \tilde{w}_1 w_2 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3}, \\
\hat{z} + \frac{\hat{\theta}_0 + \hat{\theta}_1 + \hat{\theta}_\infty}{2} = & -\frac{\tilde{w}_1 (m(\theta_0 + \tilde{m}) w_2 - \theta_0 \tilde{w}_1 \tilde{w}_3 - w_2 w_3 \tilde{w}_3)}{(m - \tilde{m}) \tilde{w}_1 w_2 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3}, \\
\hat{z} + \frac{\hat{\theta}_0 - \hat{\theta}_1 + \hat{\theta}_\infty}{2} = & -\frac{(m w_2 - \tilde{w}_1 \tilde{w}_3)((\theta_0 + m) \tilde{w}_1 + w_2 w_3)}{(m - \tilde{m}) \tilde{w}_1 w_2 + w_2^2 w_3 - \tilde{w}_1^2 \tilde{w}_3}, \quad (4.50)
\end{aligned}$$

$$\hat{y} = \frac{\tilde{w}_1 (\tilde{m}(\theta_0 + m) \tilde{w}_1 - \theta_0 w_2 w_3 - \tilde{w}_1 w_3 \tilde{w}_3)}{(\tilde{m} \tilde{w}_1 - w_2 w_3)((\theta_0 + m) \tilde{w}_1 + w_2 w_3)}, \quad (4.51)$$

$$\hat{u} = -\frac{D_{22} \mu_2 (\mu_2 + \theta_1)}{D_{11} \mu_1} \frac{(\tilde{m} \tilde{w}_1 - w_2 w_3)}{(\tilde{m}(\theta_0 + m) \tilde{w}_1 - \theta_0 w_2 w_3 - \tilde{w}_1 w_3 \tilde{w}_3)}. \quad (4.52)$$

The formulae (4.50) arise from the comparison of the different matrix elements, of course, all of them are equivalent.

We can use now the methodology of Subsection 4.3 to invert equations (4.50)–(4.52) to get a parametrization of the similarity solutions of 3WRI system in terms of “hat”  $P_5$  functions. However, there is much more sense to rewrite these equations in terms of the

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<sup>6</sup>For the convenience of the reader this parametrization is presented in equation (A.6) in Appendix A.

“uncovered”  $P_5$  functions obtained in Subsection 4.3 by exploiting equations (4.23). In this way we obtain the Okamoto-type Bäcklund transformation:

$$\hat{z} = z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}, \quad \hat{y} = \frac{yz}{z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}}, \quad \hat{u} = \frac{D_{22}\mu_2(\mu_2 + \theta_1)}{D_{11}\mu_1(z(y-1) - \theta_0)}. \quad (4.53)$$

The first two formulae here represent the Okamoto transformation for  $P_5$ . We complete them by the reference that  $y$  solves (1.3) for the coefficients (4.22), while the function  $\hat{y}$  is the solution of (1.3) for the following set of the coefficients:

$$\begin{aligned} \hat{\alpha} &= \frac{1}{2} \left( \frac{\hat{\theta}_0 - \hat{\theta}_1 + \hat{\theta}_\infty}{2} \right)^2 = \frac{\theta_1^2}{2}, \quad \hat{\beta} = -\frac{1}{2} \left( \frac{\hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_\infty}{2} \right)^2 = -\frac{\theta_0^2}{2}, \\ \hat{\gamma} &= 1 - \hat{\theta}_0 - \hat{\theta}_1 = 1 + \theta_\infty, \quad \delta = -\frac{1}{2}. \end{aligned} \quad (4.54)$$

Let us now consider the function  $\hat{u}$ . First of all, notice that we can use the formula for the logarithmic derivative of  $u$  in terms of  $y$  and  $z$  (4.21c) to get the corresponding transformation for the logarithmic derivative of  $\hat{u}$ ,

$$t \frac{d}{dt} \log \hat{u} = t \frac{d}{dt} \log u - \frac{\theta_0 - \theta_1 + \theta_\infty}{2} (y + 1) \quad (4.55)$$

The functions  $\mu_1$  and  $\mu_2$  are solutions of the quadratic equation:

$$\begin{aligned} &\mu^2 + \left( \theta_0 + \theta_1 + \frac{t}{2} \right) \mu + \frac{t\theta_1}{2} + \frac{(\theta_0 + \theta_1)^2 - \theta_\infty^2}{4} - \\ &\frac{1}{y} \left( \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) (y - 1) - \frac{\theta_0 + \theta_1 - \theta_\infty}{2} \right) \left( z(y - 1) - \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) = 0. \end{aligned}$$

Although we do not present explicit formulae for the matrix  $D$ , the logarithmic derivative of  $D_{11}/D_{22}$  or the difference  $\mathcal{D}_\infty[1, 1] - \mathcal{D}_\infty[2, 2]$  can be found explicitly by using the third equation in (4.53) and equation (4.55).

**Remark 4.2.** We note that in case we take the Fuchs–Garnier pair in Jimbo–Miwa parametrization and substitute the the functions  $y$ ,  $z$ , and  $u$ , by  $\hat{y}$ ,  $\hat{z}$ , and  $\hat{u}$ , then we get a more natural parametrization of the Fuchs–Garnier pair with the  $P_5$  functions: in this parametrization each formal monodromy is responsible for the corresponding coefficient of  $P_5$  cf. (4.54). Note that Jimbo–Miwa parameterizations for all other Painlevé equations [20] is similar to the one we are proposing in this remark, so in a sense we are proposing the “true” Jimbo–Miwa parametrization for  $P_5$ . Explicitly this parametrization is presented at the end of Appendix A

Of course, the method we use here to obtain the Okamoto transformation allows us to get the corresponding transformation for the solutions of the Fuchs–Garnier pairs. We denote as  $\hat{Y}(x, t)$  the solution of the Fuchs–Garnier pair (4.44) of Subsection 4.4 so that to make this notation consistent with the notation for  $P_5$  functions introduced above. The function  $Y(x, t)$ , in the following formula, is the solution of the Fuchs–Garnier pair (4.17), (4.18)<sup>7</sup> of Subsection 4.2. The formula relating these functions reads,

$$\hat{Y}(x, t) = x^{-\frac{m}{2}} (x - 1)^{-\frac{m}{2}} D(t)^{-1} \int_C e^{t(x - \frac{1}{2})(\tilde{x} - \frac{1}{2})} \left( P + \frac{1}{\tilde{x}} Q \right) Y(\tilde{x}, t) d\tilde{x}, \quad (4.56)$$

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<sup>7</sup>To put system (4.17), (4.18) into the standard traceless form one has to make an additional scalar gauge transformation,  $Y \rightarrow x^{\theta_0/2} (x - 1)^{1 + \theta_1/2} Y$ .

where the numbers  $\tilde{m}$  and  $m$  are defined in (4.49), matrix  $D(t)$  - in (4.43), contour  $C$  is the same as in Subsection 4.2, and

$$P = H^{-1}F, \quad Q = H^{-1} \begin{pmatrix} f_{13} \\ f_{23} \end{pmatrix} \cdot \begin{pmatrix} \frac{2}{t}B_{31} & \frac{2}{t}B_{32} \end{pmatrix},$$

$$\text{where } F \text{ is the } 2 \times 2 \text{ submatrix of } G^{-1} = \begin{pmatrix} \boxed{F} & f_{13} \\ f_{23} & * \\ * & * \end{pmatrix},$$

$G^{-1}$  is the inverse of  $G$ , defined in the beginning of this subsection, and  $B_{3k}$   $k = 1, 2$  are the matrix elements of  $B$  (4.12b). The explicit expressions for  $P$  and  $Q$  are as follows:

$$P = D_\mu \begin{pmatrix} -\frac{\tilde{m}\tilde{w}_1 - w_2w_3}{(m - \tilde{m})\tilde{w}_1w_2 + w_2^2 - \tilde{w}_1^2\tilde{w}_3} & \frac{mw_2 - \tilde{w}_1\tilde{w}_3}{(m - \tilde{m})\tilde{w}_1w_2 + w_2^2 - \tilde{w}_1^2\tilde{w}_3} \\ \frac{\tilde{m}(\theta_0 + m)\tilde{w}_1 - \theta_0w_2w_3 - \tilde{w}_1w_3\tilde{w}_3}{(m - \tilde{m})\tilde{w}_1w_2 + w_2^2 - \tilde{w}_1^2\tilde{w}_3} & -\frac{m(\theta_0 + \tilde{m})w_2 - \theta_0\tilde{w}_1\tilde{w}_3 - w_2w_3\tilde{w}_3}{(m - \tilde{m})\tilde{w}_1w_2 + w_2^2 - \tilde{w}_1^2\tilde{w}_3} \end{pmatrix},$$

$$Q = D_\mu \begin{pmatrix} 0 & 0 \\ \tilde{w}_2 & w_1 \end{pmatrix}, \quad D_\mu = \frac{1}{\mu_1 - \mu_2} \begin{pmatrix} \frac{\mu_2 + \theta_1}{\mu_1} & 0 \\ 0 & \frac{1}{\mu_2} \end{pmatrix}.$$

## 5 Similarity Reduction to the Fourth Painlevé Equation

The following similarity reduction of 3WRI system (1.10) was found in [28]:

$$u_j = e^{i\phi_j} v_j(\tau), \quad j = 1, 2, 3, \quad \tau = x_1 + x_2 + x_3, \quad (5.1)$$

where

$$\begin{aligned} \phi_1 &= \rho x_3 + \frac{1}{2}x_3^2 + 2x_2x_3 + \frac{1}{2}\rho^2, & \phi_2 &= \rho x_3 + \frac{1}{2}x_3^2 + 2x_3x_1 + \frac{1}{2}\rho^2, \\ \phi_3 &= 2\rho(x_1 + x_2) + (x_1 + x_2)^2, \end{aligned} \quad (5.2)$$

and  $\rho$  is a real constant. Under these conditions system (1.10) reduces to the system of ODEs:

$$e^{i\phi} v_1' = iv_2^* v_3^*, \quad e^{i\phi} v_2' = iv_3^* v_1^*, \quad e^{i\phi} v_3' = iv_1^* v_2^*, \quad (5.3)$$

where

$$\phi = \phi_1 + \phi_2 + \phi_3 = (\tau + \rho)^2, \quad (5.4)$$

and prime denotes differentiation with respect to  $\tau$ . This system was integrated in [28] in terms of SD-functions by splitting real and imaginary parts of the equations. These SD-functions were shown to be related with the fourth Painlevé functions (1.2).

**Remark 5.1.** As usual, it is straightforward to generalize this similarity reduction to the coupled case of the 3WRI system. One adds to (5.1) and (5.3) the formally conjugated equations:

$$u_j^* = e^{-i\phi_j} v_j^*, \quad e^{-i\phi} v_j^{*'} = -iv_k v_l,$$

respectively, where  $(j, k, l)$  is any cyclic permutation of  $(1, 2, 3)$ ,  $\phi_j$  and  $\phi$  are defined in (5.2) and (5.4), respectively, with  $\rho \in \mathbb{C}$ . As usual in the coupled case the functions  $v_j$  and  $v_j^*$  are not assumed to be complex conjugates. In the most part of this Section we deal with the coupled 3WRI system and turn back to the physical case at the end of Subsection 5.3.



### 5.1 A $3 \times 3$ Fuchs–Garnier Pair for the Reduced System

Following the approach of the previous sections, we will use (5.1) to construct a Fuchs–Garnier pair which is valid for both the coupled and physical cases of the reduced system (5.3).

Consider the Lax pair (2.2). Instead of the spectral parameter  $k$  we define the spectral parameter  $\lambda$  in the following way

$$\lambda = x_1 - x_2. \quad (5.5)$$

Since the spectral parameter is already defined we put  $\kappa_1 = \kappa_2 = \kappa_3 = 0$ , and by the direct substitution prove that  $\Psi(x_j, k) = R(x_j)\tilde{\Phi}(\tau, \lambda)$ , where  $R(x_j)$  is given by

$$R(x_1, x_2, x_3) = \text{diag} \left( e^{i\phi_2}, e^{-i\phi_1}, 1 \right).$$

In the new variables the Lax pair takes the form:

$$\begin{aligned} \tilde{\Phi}_\tau + D_1 \tilde{\Phi}_\lambda &= i \left( -(\lambda - \tau)S_2 + V_1 \right) \tilde{\Phi} \\ \tilde{\Phi}_\tau + D_2 \tilde{\Phi}_\lambda &= i \left( -(\lambda + \tau)S_1 + V_2 \right) \tilde{\Phi}, \end{aligned} \quad (5.6)$$

where the matrices  $D_j, S_j, V_j$  are given by

$$\begin{aligned} D_1 &= \text{diag}(-1, 0, 1), & D_2 &= \text{diag}(0, 1, -1), \\ S_1 &= \text{diag}(1, 0, 0), & S_2 &= \text{diag}(0, 1, 0), \\ V_1 &= \begin{pmatrix} 0 & -e^{-i\phi}v_3^* & 0 \\ 0 & \frac{1}{2}\rho & -v_1^* \\ -v_2^* & 0 & 0 \end{pmatrix}, & V_2 &= \begin{pmatrix} -\frac{1}{2}\rho & 0 & v_2 \\ e^{i\phi}v_3 & 0 & 0 \\ 0 & v_1 & 0 \end{pmatrix}. \end{aligned}$$

After rearranging, the above system can be written in the form

$$\tilde{\Phi}_\lambda = \left( \lambda Q^{(1)} + Q^{(0)} \right) \tilde{\Phi} \quad (5.7a)$$

$$\tilde{\Phi}_\tau = \left( \lambda P^{(1)} + P^{(0)} \right) \tilde{\Phi}, \quad (5.7b)$$

where the matrices  $Q^{(1)}, P^{(1)}, Q^{(0)}, P^{(0)}$  are given by

$$Q^{(1)} = -i \text{diag}(1, -1, 0), \quad P^{(1)} = -i \text{diag}(1, 1, 0),$$

and

$$Q^{(0)} = i \begin{pmatrix} -(\tau + \rho) & e^{-i\phi}v_3^* & v_2 \\ e^{i\phi}v_3 & -(\tau + \rho) & v_1^* \\ -\frac{1}{2}v_2^* & -\frac{1}{2}v_1 & 0 \end{pmatrix}, \quad P^{(0)} = \begin{pmatrix} -(\tau + \rho) & 0 & v_2 \\ 0 & (\tau + \rho) & -v_1^* \\ -\frac{1}{2}v_2^* & \frac{1}{2}v_1 & 0 \end{pmatrix}.$$

### 5.2 Fuchs–Garnier Pairs for the Fourth Painlevé Equation

We now consider the Fuchs–Garnier pair (5.7) in more detail: we introduce variables  $\{w_j, \tilde{w}_j\}$ ,  $j = 1, 2, 3$ , to emphasize the “coupled character” of the system under consideration and write

$$\Phi_\lambda = \left( \lambda B_1^4 + B_0^4 \right) \Phi \quad (5.8a)$$

$$\Phi_\tau = \left( \lambda M_1^4 + M_0^4 \right) \Phi, \quad (5.8b)$$

where the matrices  $B_1^4, M_1^4, B_0^4, M_0^4$  are given by

$$B_1^4 = -i \text{diag}(1, -1, 0), \quad M_1^4 = -i \text{diag}(1, 1, 0), \quad (5.9)$$

and

$$B_0^4 = \begin{pmatrix} -i(\tau + \rho) & \tilde{w}_3 & w_2 \\ w_3 & -i(\tau + \rho) & \tilde{w}_1 \\ \tilde{w}_2 & w_1 & 0 \end{pmatrix}, \quad M_0^4 = \begin{pmatrix} -i(\tau + \rho) & 0 & w_2 \\ 0 & i(\tau + \rho) & -\tilde{w}_1 \\ \tilde{w}_2 & -w_1 & 0 \end{pmatrix}, \quad (5.10)$$

and  $\{w_j, \tilde{w}_j\}$  are all functions of  $\tau$ . Compatibility of equations (5.8a) and (5.8b) gives the following system of equations

$$\begin{aligned} w_1' &= \tilde{w}_2 \tilde{w}_3, & \tilde{w}_1' &= -w_2 w_3, \\ w_2' &= \tilde{w}_1 \tilde{w}_3, & \tilde{w}_2' &= -w_1 w_3, \\ w_3' &= 2i(\tau + \rho)w_3 - 2\tilde{w}_1 \tilde{w}_2, & \tilde{w}_3' &= -2i(\tau + \rho)\tilde{w}_3 + 2w_1 w_2. \end{aligned} \quad (5.11)$$

Our next goal is to use the generalized Laplace transform (3.10) to construct the map between the  $3 \times 3$  Fuchs–Garnier pair (5.8) and the one in  $2 \times 2$  matrices for  $P_4$  found by Jimbo and Miwa [20].

Substituting the formula for  $\Phi(\lambda, \tau)$  from equation (3.10) into equations (5.8a) and (5.8b), and assuming that the contour  $C$  is suitably chosen to eliminate any remainder terms that arise from integration-by-parts, we find

$$B_1^4 \frac{d\tilde{Y}}{dx} = (-xI + B_0^4)\tilde{Y}, \quad (5.12)$$

$$\frac{d\tilde{Y}}{dt} = (ixB_1^4 + M_0^4 - iB_1^4 B_0^4)\tilde{Y}, \quad (5.13)$$

where for the derivation of equation (5.13) we used the identity  $i(B_1^4)^2 = M_1^4$  (see equations (5.9)) and the matrices  $B_0^4, M_0^4$  are given in (5.10). We note that, because the diagonal matrix  $B_1^4$  has a zero in the (33) entry, the third rows of these equations give the following relationship between the components of  $\tilde{Y}$

$$x\tilde{Y}_3 = \tilde{w}_2 \tilde{Y}_1 + w_1 \tilde{Y}_2, \quad \frac{d}{dt}\tilde{Y}_3 = \tilde{w}_2 \tilde{Y}_1 - w_1 \tilde{Y}_2.$$

Using the first equation above to eliminate  $\tilde{Y}_3$  from equations (5.12) and (5.13) we arrive at the following  $2 \times 2$  system

$$\begin{aligned} \frac{d\hat{Y}}{dx} &= \left( xA_2^4 + A_1^4(\tau) + \frac{A_0^4(\tau)}{x} \right) \hat{Y}, \\ \frac{d\hat{Y}}{dt} &= (xA_2^4 + A_1^4(\tau) - i(\tau + \rho)A_2^4) \hat{Y}, \quad \hat{Y} = \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix}, \end{aligned} \quad (5.14)$$

where

$$A_0^4(\tau) = \begin{pmatrix} -w_2 \tilde{w}_2 & -w_2 w_1 \\ \tilde{w}_2 \tilde{w}_1 & w_1 \tilde{w}_1 \end{pmatrix}, \quad A_1^4(\tau) = \begin{pmatrix} i(\tau + \rho) & -\tilde{w}_3 \\ w_3 & -i(\tau + \rho) \end{pmatrix}, \quad A_2^4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We remark that system (5.14) is related to the Jimbo–Miwa system for  $P_4$  (see equations (C.30) and (C.31) of [20]) by a simple gauge transformation and a change of variables.

### 5.3 Parametrization of Solutions in Terms of $P_4$

We begin with a parametrization of the general solution of system (5.11) by the (general) solution of  $P_4$ . For this purpose we use the correspondence between the  $3 \times 3$  and  $2 \times 2$  Fuchs–Garnier representations for this system.

First of all we notice that system (5.11) admits first integrals:

$$w_1 \tilde{w}_1 - w_2 \tilde{w}_2 = 2i\theta_0, \quad w_1 \tilde{w}_1 + w_2 \tilde{w}_2 + w_3 \tilde{w}_3 = 2i\theta_\infty, \quad (5.15)$$

where  $\theta_0$  and  $\theta_\infty$  are constants, which have a sense of formal monodromies of the solution  $\hat{Y}$  of system (5.14). Motivated by the parametrization used by Jimbo–Miwa (see equations (5.14) above) we define the functions

$$\tilde{y}(\tau) = -\frac{2w_1w_2}{\tilde{w}_3} \quad \text{and} \quad \tilde{z}(\tau) = w_1\tilde{w}_1. \quad (5.16)$$

Now from (5.11) we find that  $\tilde{y}$  and  $\tilde{z}$  satisfy the following system of nonlinear ODEs:

$$\frac{d\tilde{y}}{d\tau} = -4\tilde{z} + \tilde{y}^2 + 2i(\tau + \rho)\tilde{y} + 4i\theta_0, \quad \frac{d\tilde{z}}{d\tau} = -\tilde{y}(\tilde{z} - i(\theta_0 + \theta_\infty)) - \frac{2}{\tilde{y}}\tilde{z}(\tilde{z} - 2i\theta_0). \quad (5.17)$$

Eliminating from this system the function  $\tilde{z}(\tau)$  one finds the following second order ODE for  $\tilde{y}(\tau)$ ,

$$\frac{d^2\tilde{y}}{d\tau^2} = \frac{1}{2\tilde{y}}\left(\frac{d\tilde{y}}{d\tau}\right)^2 + \frac{3}{2}\tilde{y}^3 + 4i(\tau + \rho)\tilde{y}^2 + 2(-(\tau + \rho)^2 + i(1 - 2\theta_\infty))\tilde{y} + \frac{8\theta_0^2}{\tilde{y}}.$$

We note that, under the change of variables  $(\tau + \rho) \mapsto e^{-i\pi/4}t$ ,  $\tilde{y} \mapsto e^{i\pi/4}y$ , this equation is mapped to the  $P_4$  equation (1.2) with  $\alpha = 2\theta_\infty - 1$  and  $\beta = -8\theta_0^2$ .

By using the parametrization for  $\tilde{y}(\tau)$  and  $\tilde{z}(\tau)$  given above we obtain the following expressions for the functions  $\{w_j(\tau), \tilde{w}_j(\tau)\}$ :

$$\begin{aligned} w_1 &= -\frac{f\tilde{y}\tilde{z}}{2}, & w_2 &= \frac{1}{g\tilde{z}}, & w_3 &= -\frac{2g}{f}(\tilde{z} - i\theta_0 - i\theta_\infty), \\ \tilde{w}_1 &= -\frac{2}{f\tilde{y}}, & \tilde{w}_2 &= g\tilde{z}(\tilde{z} - 2i\theta_0), & \tilde{w}_3 &= \frac{f}{g}, \end{aligned} \quad (5.18)$$

where  $f(\tau)$  and  $g(\tau)$  satisfy the equations

$$\begin{aligned} \frac{d}{d\tau} \log f &= -\frac{\tilde{y}'}{\tilde{y}} - \frac{1}{2} \left( \frac{\tilde{z}'}{\tilde{z}} - \frac{\tilde{y}(\tilde{z} - i\theta_0 - i\theta_\infty)}{\tilde{z}} + \frac{2(\tilde{z} - 2i\theta_0)}{\tilde{y}} \right), \\ \frac{d}{d\tau} \log g &= -\frac{\tilde{z}'}{\tilde{z}} - \frac{1}{2} \left( \frac{\tilde{z}'}{\tilde{z} - 2i\theta_0} - \frac{\tilde{y}(\tilde{z} - i\theta_0 - i\theta_\infty)}{\tilde{z} - 2i\theta_0} + \frac{2\tilde{z}}{\tilde{y}} \right). \end{aligned} \quad (5.19)$$

Integrating these equations we get:

$$\begin{aligned} \frac{2}{f\tilde{y}} &= (2\tilde{z})^{1/2} \exp \left( \int_{\tau_0}^{\tau} \left( \frac{\tilde{z} - 2i\theta_0}{\tilde{y}} - \frac{\tilde{y}}{2\tilde{z}}(\tilde{z} - i\theta_0 - i\theta_\infty) \right) d\tau + i\tilde{f}_0 \right), \\ \frac{1}{g\tilde{z}} &= (2(\tilde{z} - 2i\theta_0))^{1/2} \exp \left( \int_{\tau_0}^{\tau} \left( -\frac{\tilde{z}}{\tilde{y}} + \frac{\tilde{y}(\tilde{z} - i\theta_0 - i\theta_\infty)}{2(\tilde{z} - 2i\theta_0)} \right) d\tau + i\tilde{g}_0 \right), \end{aligned} \quad (5.20)$$

where  $\tau_0, \tilde{f}_0, \tilde{g}_0$  are constants of integration. In fact, in general we need only two parameters, say  $f_0$  and  $g_0$ , while  $t_0$  can be fixed. So, formulae (5.17)-(5.20) solve the problem of parametrization of general complex solutions of system (5.11) in terms of general complex solutions of  $P_4$ .

The formulae for the functions  $f(\tau)$  and  $g(\tau)$  can be simplified by introducing the function  $\tilde{\sigma}(\tau)$  following the work of Jimbo–Miwa [20]. We use  $\tilde{\sigma}(\tau)$  to eliminate the dependence on  $\tilde{y}$ . The latter, as we see below, is also important for specification of the physical solutions of the 3WRI system.

Consider the following identity, which can be proved just by differentiation with the help of (5.11),

$$w_1w_2w_3 + \tilde{w}_1\tilde{w}_2\tilde{w}_3 + i(\tau + \rho)(w_1\tilde{w}_1 + w_2\tilde{w}_2) = i \int_{\tau_0}^{\tau} (w_1\tilde{w}_1 + w_2\tilde{w}_2) d\tau.$$

Substituting in the expressions for  $\{w_j, \tilde{w}_j\}$  given in (5.18) we find

$$\tilde{y}(\tilde{z} - i\theta_0 - i\theta_\infty) - \frac{2\tilde{z}(\tilde{z} - 2i\theta_0)}{\tilde{y}} = -2i(\tau + \rho)(\tilde{z} - i\theta_0) + 2i \int_{\tau_0}^{\tau} (\tilde{z} - i\theta_0) d\tau.$$

We define the function  $\tilde{\sigma}(\tau)$  (cf. [20]) by

$$i\tilde{\sigma} = -\tilde{y}(\tilde{z} - i\theta_0 - i\theta_\infty) + \frac{2\tilde{z}(\tilde{z} - 2i\theta_0)}{\tilde{y}} - 2i(\tau + \rho)\tilde{z}. \quad (5.21)$$

It follows from the above identity that we have for the derivative of  $\tilde{\sigma}(\tau)$ ,

$$\tilde{\sigma}' = -2\tilde{z}. \quad (5.22)$$

Summing up and subtracting definition (5.21) with the second equation (5.17) and solving the result with respect to  $\tilde{y}$  and  $1/\tilde{y}$ , respectively, one finds:

$$\tilde{y} = -\frac{\tilde{z}' + i\tilde{\sigma} + 2i(\tau + \rho)\tilde{z}}{2(\tilde{z} - i\theta_0 - i\theta_\infty)}, \quad \frac{1}{\tilde{y}} = -\frac{\tilde{z}' - i\tilde{\sigma} - 2i(\tau + \rho)\tilde{z}}{4\tilde{z}(\tilde{z} - 2i\theta_0)}. \quad (5.23)$$

The compatibility condition of system (5.23) with the help of (5.22) is equivalent to the following SD-equation for the function  $\tilde{\sigma}(\tau)$ ,

$$\left(\frac{d^2\tilde{\sigma}}{d\tau^2}\right)^2 = -4\left((\tau + \rho)\frac{d\tilde{\sigma}}{d\tau} - \tilde{\sigma}\right)^2 - 4\frac{d\tilde{\sigma}}{d\tau}\left(\frac{d\tilde{\sigma}}{d\tau} + 4i\theta_0\right)\left(\frac{d\tilde{\sigma}}{d\tau} + 2i\theta_0 + 2i\theta_\infty\right). \quad (5.24)$$

With the help of the function  $\tilde{\sigma}(\tau)$  equations (5.20) can be rewritten as follows:

$$\begin{aligned} \frac{2}{f\tilde{y}} &= \sqrt{2\tilde{z}} \exp\left(i\frac{(\tau + \rho)^2}{2} + i \int_{\tau_0}^{\tau} \frac{\tilde{\sigma}}{2\tilde{z}} d\tau + if_0\right), \\ \frac{1}{g\tilde{z}} &= \sqrt{2(\tilde{z} - 2i\theta_0)} \exp\left(-i\frac{(\tau + \rho)^2}{2} - i \int_{\tau_0}^{\tau} \frac{\tilde{\sigma} + 4i\theta_0(\tau + \rho)}{2(\tilde{z} - 2i\theta_0)} d\tau - ig_0\right). \end{aligned} \quad (5.25)$$

It is important to notice that formulae (5.25), (5.23), and (5.22), allow one to parameterize the functions  $w_j$  and  $\tilde{w}_j$  for  $j = 1, 2, 3$  in terms of one function  $\tilde{\sigma}$ . We call it  $\sigma$ -parametrization.

We are now ready to solve the reduced 3WRI system (5.3) in terms of  $P_4$ . Comparing the linear systems (5.7) and (5.8) we get the following correspondence

$$v_1(\tau) = 2iw_1(\tau), \quad v_2(\tau) = -iw_2(\tau), \quad v_3(\tau) = -ie^{-i(\tau+\rho)^2}w_3(\tau), \quad (5.26a)$$

$$v_1^*(\tau) = -i\tilde{w}_1(\tau), \quad v_2^*(\tau) = 2i\tilde{w}_2(\tau), \quad v_3^*(\tau) = -ie^{i(\tau+\rho)^2}\tilde{w}_3(\tau), \quad (5.26b)$$

where the functions  $\{w_j, \tilde{w}_j\}$  are given in terms of  $\tilde{y}$  and  $\tilde{z}$  by equations (5.18) and (5.20). This provides us the general similarity solution for the coupled 3WRI system in terms of  $P_4$ .

In the physical situation we have to find the solution for which  $v_j = v_j^*$ , where now the  $*$  means complex conjugation. In this case we impose the following restrictions on the parameters:

$$\rho, t, t_0, f_0, g_0 \in \mathbb{R} \quad \text{and} \quad \theta_0, \theta_\infty \in i\mathbb{R}.$$

In that case the equation for  $\tilde{\sigma}(\tau)$ , equation (5.24), has real coefficients and we can take its general real solution. Then using the  $\sigma$ -parametrization of the functions  $w_j$  and  $\tilde{w}_j$  one proves that the physical reduction is indeed fulfilled, provided  $\tilde{z} > 0$  and  $\tilde{z} > 2i\theta_0$ .

#### 5.4 Relation between the Noumi–Yamada and Jimbo–Miwa Fuchs–Garnier Pairs for the Fourth Painlevé Equation

The symmetric form of  $P_4$  [6, 40, 1, 32] reads:

$$\begin{aligned}\frac{df_0}{dz} &= f_0(f_1 - f_2) + \alpha_0, \\ \frac{df_1}{dz} &= f_1(f_2 - f_0) + \alpha_1, \\ \frac{df_2}{dz} &= f_2(f_0 - f_1) + \alpha_2,\end{aligned}\tag{5.27}$$

where  $\alpha_k \in \mathbb{C}$  satisfy the following normalization condition,

$$\alpha_0 + \alpha_1 + \alpha_2 = 1.$$

The functions

$$y_k(t) = \sqrt{-2}f_k(z), \quad t = z/\sqrt{-2}, \quad k = 0, 1, 2\tag{5.28}$$

solve  $P_4$  (1.2) for

$$\alpha = \alpha_{k+1(\bmod 3)} - \alpha_{k+2(\bmod 3)}, \quad \beta = -2\alpha_k^2.\tag{5.29}$$

The following Fuchs–Garnier pair in  $3 \times 3$  matrices was obtained for (5.27) by Noumi and Yamada [33],

$$\frac{d\Psi}{d\mu} = - \left( \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ f_0 & 1 & 0 \end{pmatrix} + \frac{1}{\mu} \begin{pmatrix} v_1 & f_1 & 1 \\ 0 & v_2 & f_2 \\ 0 & 0 & v_3 \end{pmatrix} \right) \Psi \equiv -(A + \frac{1}{\mu}B)\Psi,\tag{5.30}$$

$$\frac{d\Psi}{dz} = - \left( \mu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} \frac{z}{3} - f_2 & 1 & 0 \\ 0 & \frac{z}{3} - f_0 & 1 \\ 0 & 0 & \frac{z}{3} - f_1 \end{pmatrix} \right) \Psi = -(\mu P + Q)\Psi,\tag{5.31}$$

where the numbers  $v_k$  and  $\alpha_k$  are related as follows:

$$\alpha_0 = 1 + v_3 - v_1, \quad \alpha_1 = v_1 - v_2, \quad \alpha_2 = v_2 - v_3.$$

Note that, for a given set of the numbers  $\{\alpha_k\}_{k=0,1,2}$ , any one of the numbers  $v_k$  can be taken arbitrarily. Using this fact we assume, without loss of generality, that

$$v_1 = 1.\tag{5.32}$$

This assumption is equivalent to the additional transformation,  $\Psi \rightarrow \mu^{1+v_1}\Psi$ .

Now making for the solution  $\Psi$  the generalized Laplace transform

$$\Psi(\mu, z) = \int_C e^{\mu\zeta} \tilde{\Psi}(\zeta, z) d\zeta$$

with the suitably chosen contour  $C$  (such that the corresponding terms appearing due to integration by parts cancel) we arrive at the following system of equations for the Laplace image  $\tilde{\Psi}$ :

$$\begin{aligned}\frac{d\tilde{\Psi}}{d\zeta} &= (\zeta I + A)^{-1}(B - I)\tilde{\Psi}, \\ \frac{d\tilde{\Psi}}{dz} &= (P(\zeta I + A)^{-1}(B - I) - Q)\tilde{\Psi},\end{aligned}\tag{5.33}$$

In view of the condition (5.32) the solution of system (5.33) is reduced to the following system in  $2 \times 2$  matrices for the column vector  $\tilde{Y} \equiv \tilde{Y}(x, z) = (\tilde{\Psi}_2(\zeta, z), \tilde{\Psi}_3(\zeta, z))^T$ , where  $x = 1/\zeta$  and  $\tilde{\Psi}_2, \tilde{\Psi}_3$  are the components of  $\tilde{\Psi} = (\tilde{\Psi}_1, \tilde{\Psi}_2, \tilde{\Psi}_3)^T$ :

$$\begin{aligned} \frac{d\tilde{Y}}{dx} &= \left( \begin{pmatrix} 0 & 0 \\ -f_1 & -1 \end{pmatrix} x + \begin{pmatrix} f_1 & 1 \\ f_0 f_1 + v_2 - 1 & f_0 + f_2 \end{pmatrix} + \frac{1}{x} \begin{pmatrix} 1 - v_2 & -f_2 \\ 0 & 1 - v_3 \end{pmatrix} \right) \tilde{Y}, \\ \frac{d\tilde{Y}}{dz} &= \left( \begin{pmatrix} 0 & 0 \\ -f_1 & -1 \end{pmatrix} x + \begin{pmatrix} f_0 - \frac{z}{3} & -1 \\ 0 & f_1 - \frac{z}{3} \end{pmatrix} \right) \tilde{Y}. \end{aligned} \quad (5.34)$$

System (5.34) is mapped to the standard  $2 \times 2$  Jimbo–Miwa Fuchs–Garnier pair for  $P_4$  (equations (C.30), (C.31)) of [20] by an appropriate triangular gauge transformation and rescaling the variables  $x \mapsto \sqrt{-2}x$ ,  $z = \sqrt{-2}t$ . The resulting system is parameterized in terms of the function  $y_2(t)$ , defined by (5.28) for  $k = 2$ , which solves  $P_4$  with coefficients (5.29).

We conclude this subsection by noting that all of the transformations presented here and in Subsection 5.2 are invertible. Thus it is straightforward to construct a direct mapping between the  $3 \times 3$  Fuchs–Garnier pair for  $P_4$  obtained in this paper (5.8) and the one by Noumi and Yamada for the symmetric form of  $P_4$  (5.30).

## 6 Similarity Reduction to the Third Painlevé Equation

The following similarity reduction was derived independently in [25] and [28]:

$$u_1 = \exp[\frac{1}{2}x_3 + \frac{i}{2}\rho x_3]v_1, \quad u_2 = \exp[\frac{1}{2}x_3 + \frac{i}{2}\rho x_3]v_2, \quad u_3 = (x_1 - x_2)^{-1+i\rho}v_3, \quad (6.1)$$

where  $v_j = v_j(\tau)$  with

$$\tau = (x_1 - x_2)e^{x_3}, \quad (6.2)$$

and  $\rho$  is a real constant. In this case system (1.10) reduces to the system of ODEs:

$$\tau^{1+i\rho}v_1' = iv_2^*v_3^*, \quad \tau^{1+i\rho}v_2' = -iv_3^*v_1^*, \quad \tau^{i\rho}v_3' = iv_1^*v_2^*. \quad (6.3)$$

It was shown in [28] that solutions of this system can be represented in terms of an SD-type equation that is equivalent to the particular case of the fifth Painlevé equation (1.3) with  $\delta = 0$ . It is well known [16] that equation (1.3) with  $\delta$  taken to be zero is equivalent to the general  $P_3$  equation.

**Remark 6.1.** As usual we generalize this similarity reduction to the coupled case of the 3WRI system. To do it one adds to (6.1) and (6.3) the formally conjugated equations

$$\begin{aligned} u_1^* &= \exp[\frac{1}{2}x_3 - \frac{i}{2}\rho x_3]v_1^*, \quad u_2^* = \exp[\frac{1}{2}x_3 + \frac{i}{2}\rho x_3]v_2^*, \quad u_3^* = (x_1 - x_2)^{-1-i\rho}v_3, \\ \tau^{1-i\rho}v_1^{*'} &= -iv_2v_3, \quad \tau^{1-i\rho}v_2^{*'} = +iv_3v_1, \quad \tau^{-i\rho}v_3^{*'} = -iv_1v_2. \end{aligned}$$

Note that in the coupled case  $\rho \in \mathbb{C}$  and the functions  $v_j$  and  $v_j^*$  are not assumed to be complex conjugates. In the most part of this Section we deal with the coupled 3WRI system and turn back to the physical case at the end of Subsection 6.3.

### 6.1 A $3 \times 3$ Fuchs–Garnier Pair for the Reduced System

As in the case of  $P_4$ ,  $P_5$  and  $P_6$  we will construct a Fuchs–Garnier pair for the reduced system (6.3) and then carry out the explicit integration in terms of  $P_3$ . We introduce the spectral parameter  $\lambda$  as

$$\lambda = e^{-x_3}k. \quad (6.4)$$

Taking  $\kappa_3 = 0$  in (2.2) and writing  $\Psi(x_j, k) = R(x_j)\Phi(\tau, \lambda)$  where  $R(x_j)$  is given by

$$R(x_1, x_2, x_3) = \text{diag} \left( \exp[\tfrac{i}{2}\rho x_3], \exp[-\tfrac{i}{2}\rho x_3], \exp[-\tfrac{1}{2}x_3] \right),$$

we obtain the following Fuchs–Garnier pair

$$M\Phi_\lambda = \left( Q^{(1)} + \frac{Q^{(0)}}{\lambda} \right) \Phi \quad (6.5a)$$

$$\Phi_\tau = \left( \lambda P^{(1)} + P^{(0)} \right) \Phi, \quad (6.5b)$$

where the matrices  $M, Q^{(1)}, P^{(1)}, Q^{(0)}, P^{(0)}$  are given by

$$M = \text{diag} (1, 1, 0),$$

and

$$Q^{(1)} = i \text{diag} (-\tau\kappa_2, \tau\kappa_1, \kappa_1 + \kappa_2), \quad P^{(1)} = i \text{diag} (-\kappa_2, \kappa_1, 0),$$

$$Q^{(0)} = i \begin{pmatrix} \frac{1}{2}\rho & \tau^{-i\rho}v_3^* & -v_2 \\ \tau^{i\rho}v_3 & -\frac{1}{2}\rho & v_1^* \\ -v_2^* & v_1 & 0 \end{pmatrix}, \quad P^{(0)} = i \begin{pmatrix} 0 & \tau^{-1-i\rho}v_3^* & 0 \\ \tau^{-1+i\rho}v_3 & 0 & 0 \\ -\frac{1}{2}v_2^* & -\frac{1}{2}v_1 & 0 \end{pmatrix}.$$

## 6.2 Fuchs–Garnier Pairs for the Third Painlevé Equation

As in the previous sections, to avoid a confusion with the complex conjugation we introduce the following simpler notation:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Phi_\lambda = \left( \begin{pmatrix} \tau/2 & 0 & 0 \\ 0 & -\tau/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} -\theta_\infty/2 & -\tilde{w}_3 & -w_2 \\ w_3 & \theta_\infty/2 & -\tilde{w}_1 \\ \tilde{w}_2 & w_1 & 0 \end{pmatrix} \right) \Phi \quad (6.6a)$$

$$\Phi_\tau = \left( \begin{pmatrix} \lambda/2 & 0 & 0 \\ 0 & -\lambda/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{\tau} \begin{pmatrix} 0 & -\tilde{w}_3 & 0 \\ w_3 & 0 & 0 \\ \frac{\tau}{2}\tilde{w}_2 & -\frac{\tau}{2}w_1 & 0 \end{pmatrix} \right) \Phi, \quad (6.6b)$$

where  $\{w_j, \tilde{w}_j\}$  are functions of  $\tau$  and  $\theta_\infty$  is an arbitrary constant. The compatibility condition for (6.6) is

$$\begin{aligned} \tau w'_1 &= \tilde{w}_2 \tilde{w}_3, & \tau \tilde{w}'_1 &= w_2 w_3, \\ \tau w'_2 &= -\tilde{w}_1 \tilde{w}_3, & \tau \tilde{w}'_2 &= -w_1 w_3, \\ \tau w'_3 &= -\theta_\infty w_3 + \tau \tilde{w}_1 \tilde{w}_2, & \tau \tilde{w}'_3 &= \theta_\infty \tilde{w}_3 + \tau w_1 w_2. \end{aligned}$$

We note that the third row of (6.6a) gives the relation

$$\lambda \Phi_3 = \tilde{w}_2 \Phi_1 + w_1 \Phi_2,$$

and so we can eliminate  $\Phi_3$  from the above system. The resulting  $2 \times 2$  system has the form:

$$\frac{d\phi}{d\lambda} = \left( \frac{\tau}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} -\theta_\infty/2 & -\tilde{w}_3 \\ w_3 & \theta_\infty/2 \end{pmatrix} - \frac{1}{\lambda^2} \begin{pmatrix} w_2 \tilde{w}_2 & w_1 w_2 \\ \tilde{w}_1 \tilde{w}_2 & w_1 \tilde{w}_1 \end{pmatrix} \right) \phi, \quad (6.7a)$$

$$\frac{d\phi}{d\tau} = \left( \frac{\lambda}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{\tau} \begin{pmatrix} 0 & -\tilde{w}_3 \\ w_3 & 0 \end{pmatrix} \right) \phi, \quad \phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \quad (6.7b)$$

Making the change of variables  $\tau \mapsto t^2, \lambda \mapsto \lambda/t, w_j(\tau) \mapsto W_j(t)$ , we find that system (6.7) is equivalent (up to a diagonal gauge transformation) to the Jimbo–Miwa system for the complete  $P_3$  [20] in case  $w_1 \tilde{w}_1 + w_2 \tilde{w}_2 = c_1 \neq 0$ , or to the degenerate case of  $P_3$  (see e.g. [26]) otherwise. Below we present the corresponding parametrization for both cases.

### 6.3 Parametrization of solutions in terms of $P_3$

Written in terms of the new variables the compatibility condition for (6.6) becomes

$$\begin{aligned} tW_1' &= 2\tilde{W}_2\tilde{W}_3, & t\tilde{W}_1' &= 2W_2W_3, \\ tW_2' &= -2\tilde{W}_1\tilde{W}_3, & t\tilde{W}_2' &= -2W_1W_3, \\ tW_3' &= -2\theta_\infty W_3 + 2t^2\tilde{W}_1\tilde{W}_2, & t\tilde{W}_3' &= 2\theta_\infty\tilde{W}_3 + 2t^2W_1W_2, \end{aligned} \quad (6.8)$$

where  $\{W_j, \tilde{W}_j\}$  are functions of  $t$ . This system admits the following first integrals

$$\begin{aligned} W_1\tilde{W}_1 + W_2\tilde{W}_2 &= c_1, \\ W_1W_2W_3 - \tilde{W}_1\tilde{W}_2\tilde{W}_3 + \frac{\theta_\infty}{2}(W_1\tilde{W}_1 - W_2\tilde{W}_2) &= \frac{\theta_0}{2}, \end{aligned} \quad (6.9)$$

where  $c_1$  and  $\theta_0$  are constants. Elementary computation now shows that the function  $y$  given by

$$y(t) = \frac{\tilde{W}_3}{tW_1W_2}, \quad (6.10)$$

satisfies the  $P_3$  equation (1.1) with  $\alpha = 4\theta_0$ ,  $\beta = 4(1 - \theta_\infty)$ ,  $\gamma = 4c_1^2$ ,  $\delta = -4$ . To parameterize the functions  $\{W_j(t), \tilde{W}_j(t)\}$  in terms of  $P_3$ , we introduce the functions:

$$z = tW_1\tilde{W}_1, \quad w = W_1W_2.$$

Note that our notation for  $w(t)$  is slightly different from the one taken by Jimbo and Miwa [20]. Using the expression for  $y$  given in (6.10) and the compatibility conditions (6.8), we obtain the following system for  $\{y, z, w\}$

$$\begin{aligned} t\frac{dy}{dt} &= 2(2z - c_1t)y^2 + (2\theta_\infty - 1)y + 2t, \\ t\frac{dz}{dt} &= 4z(c_1t - z)y - (2\theta_\infty - 1)z + (\theta_0 + c_1\theta_\infty)t, \\ t\frac{d}{dt}(\ln w) &= 2(c_1t - 2z)y. \end{aligned}$$

It follows from (6.8) and the above expressions that the functions  $\{W_j, \tilde{W}_j\}$  are given as

$$\begin{aligned} W_1(t) &= \frac{zf}{c_1t - z}, & W_2(t) &= g\frac{c_1t - z}{z}, \\ \tilde{W}_1(t) &= \frac{c_1t - z}{tf}, & \tilde{W}_2(t) &= \frac{z}{tg}, \\ W_3(t) &= \frac{1}{tfg} \left( yz(c_1t - z) - \theta_\infty z + \frac{\theta_0 + c_1\theta_\infty}{2}t \right), \\ \tilde{W}_3(t) &= tfgy, \end{aligned} \quad (6.11)$$

where the functions  $f(t)$  and  $g(t)$  satisfy the following equations

$$\begin{aligned} t\frac{d}{dt} \log f &= 2y(c_1t - z) - t\frac{d}{dt} \log \left( \frac{z}{c_1t - z} \right), \\ t\frac{d}{dt} \log g &= -2yz - t\frac{d}{dt} \log \left( \frac{c_1t - z}{z} \right). \end{aligned} \quad (6.12)$$

Now we are ready to solve the reduced 3WRI system (6.3) in terms of  $P_3$ . Just comparing systems (6.5) and (6.6) we find:

$$\begin{aligned} v_1(\tau) &= -iW_1(t), & v_2(\tau) &= -iW_2(t), & v_3(\tau) &= -it^{-2i\rho}W_3(t), & t &= \sqrt{\tau}, \\ v_1^*(\tau) &= i\tilde{W}_1(t), & v_2^*(\tau) &= i\tilde{W}_2(t), & v_3^*(\tau) &= it^{2i\rho}\tilde{W}_3(t), & \rho &= i\theta_\infty, \end{aligned} \quad (6.13)$$



where the functions  $W_j(t)$  are given by equations (6.11). Equations (6.13) provide a solution of the coupled 3WRI system.

To get the solutions for the physical case we have to guarantee that  $v_j$  and  $v_j^*$  are indeed complex conjugates. In order to do this we notice that equations (6.12) imply the following formulae for the functions  $f$  and  $g$ :

$$\begin{aligned} f(t) &= \frac{c_1 t - z}{z} \sqrt{\frac{z}{t}} |t|^{\theta_\infty} \exp \left( -\frac{\theta_0 + c_1 \theta_\infty}{2} \int_{t_0}^t \frac{dt}{z} + i c_2 \right), \\ g(t) &= \frac{z}{\sqrt{t} \sqrt{c_1 t - z}} |t|^{\theta_\infty} \exp \left( \frac{\theta_0 - c_1 \theta_\infty}{2} \int_{t_0}^t \frac{dt}{c_1 t - z} + i c_3 \right), \end{aligned}$$

where  $t_0, c_2, c_3 \in \mathbb{R}$  and the function  $z(t)$  solves the following ODE,

$$\begin{aligned} \frac{d^2 z}{dt^2} &= \frac{c_1 t - 2z}{2z(c_1 t - z)} \left( \frac{dz}{dt} \right)^2 + \frac{z}{t(c_1 t - z)} \left( \frac{dz}{dt} \right) + \frac{8z(c_1 t - z)}{t} + \frac{c_1 + 4\theta_0 \theta_\infty}{2t} \\ &\quad + \frac{(\theta_0 - c_1 \theta_\infty)^2 - c_1^2}{2(c_1 t - z)} - \frac{(\theta_0 + c_1 \theta_\infty)^2}{2z}. \end{aligned} \quad (6.14)$$

Therefore, if we choose  $c_1 \in \mathbb{R}$  and  $\theta_0, \theta_\infty \in i\mathbb{R}$ , then the function  $z(t)$  can be taken real for real  $t$  and should satisfy the following inequalities,  $0 < z(t)/t < c_1$ . It is now easy to observe that under these conditions equations (6.13) define a similarity solution for the physical reduction of 3WRI system.

In the case  $c_1 \neq 0$  the function  $y(\tilde{t})$  defined by the following change of variables:

$$z(t) = c_1 t \frac{y(\tau)}{y(\tau) - 1}, \quad \tau = t^2,$$

solves the degenerate case of the  $P_5$  equation (1.3) with the coefficients:

$$\alpha = \frac{(\theta_0 - c_1 \theta_\infty)^2}{8c_1^2}, \quad \beta = -\frac{(\theta_0 + c_1 \theta_\infty)^2}{8c_1^2}, \quad \gamma = 2c_1, \quad \delta = 0.$$

In the special case  $c_1 = 0$  equation (6.14) coincides,  $z(t) = y(t)$ , with the degenerate case of the  $P_3$  equation (1.1) with the coefficients:

$$\alpha = -8, \quad \beta = 2\theta_0 \theta_\infty, \quad \gamma = 0, \quad \delta = -\theta_0^2.$$

## 6.4 Alternate Fuchs–Garnier Pairs for the Third Painlevé Equation

To conclude this section we state without proof an alternate reduction of system (6.6) to a  $2 \times 2$  system. Using the generalized Laplace transform (3.10) in (6.6a), the resulting matrix equation has the form

$$\begin{pmatrix} x - t/2 & 0 & 0 \\ 0 & x + t/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{d\tilde{Y}}{dx} = - \begin{pmatrix} -\theta_\infty/2 + 1 & -\tilde{w}_3 & -w_2 \\ w_3 & \theta_\infty/2 + 1 & -\tilde{w}_1 \\ \tilde{w}_2 & w_1 & 0 \end{pmatrix} \tilde{Y}. \quad (6.15)$$

The determinant of the r.h.s. matrix is zero by our choice of first integrals for the  $\{w_j, \tilde{w}_j\}$  system: by a special choice of the parameter  $c_1$  in (6.9)<sup>8</sup>. We can therefore make a gauge transformation  $\tilde{Y} = G\hat{Y}$  where  $G$  is the diagonalizing matrix, to obtain

$$\frac{d\hat{Y}}{dx} = \left( \hat{A}_2 + \frac{1}{x - t/2} \hat{A}_1 + \frac{1}{x + t/2} \hat{A}_0 \right) \hat{Y}, \quad (6.16)$$

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<sup>8</sup>Recall that  $w_j(\tau) = W_j(t)$ , see the end of Subsection 6.2.

where the  $\hat{A}_j$  are all of the form

$$\hat{A}_j = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 0 \end{pmatrix}.$$

Equation (6.16) can be reduced (after a change of variables) to a  $2 \times 2$  system of the form

$$\frac{dY}{dx} = \left( A_2 + \frac{1}{x-t} A_1 + \frac{1}{x} A_0 \right) Y, \quad (6.17)$$

where  $\text{tr} A_2 = \det A_2 = 0$ . Isomonodromy deformations in  $t$  of this equation are parameterized by solutions of the degenerate fifth Painlevé equation (1.3) with  $\delta = 0$ , see [27], [22]. It is interesting that in [27] was found a quadratic transformation relating isomonodromy deformations of equations (6.7) and (6.16) for special values of the corresponding formal monodromies. The results of this section allow one to find a different transformation between these equations and corresponding isomonodromy deformations for *all values* of formal monodromies. We are going to present the details of this correspondence in a separate work.

**Acknowledgements** This work was supported by ARC grant #DP0559019. The research was carried out during AVK's visits to the School of Mathematics and Statistics at the University of Sydney, Australia. We are also grateful to the referee for valuable comments which helped us to improve the paper.

## A On Spectral Interpretation of the Bäcklund Transformations for $P_5$

We recall the Bäcklund transformation for  $P_5$  that was found in [16] (see also [17]):

$$\hat{y} = 1 - \frac{2\sqrt{-2\delta}ty}{ty' - \sqrt{2\alpha}y^2 + (\sqrt{2\alpha} - \sqrt{-2\beta} + t\sqrt{-2\delta})y + \sqrt{-2\beta}}, \quad (A.1)$$

$$\sqrt{2\hat{\alpha}} = \frac{1}{2} \left( \frac{\gamma}{\sqrt{-2\delta}} + 1 - \sqrt{-2\beta} - \sqrt{2\alpha} \right), \quad (A.2)$$

$$\sqrt{-2\hat{\beta}} = \frac{1}{2} \left( \frac{\gamma}{\sqrt{-2\delta}} - 1 + \sqrt{-2\beta} + \sqrt{2\alpha} \right), \quad (A.3)$$

$$\frac{\hat{\gamma}}{\sqrt{-2\hat{\delta}}} = \sqrt{-2\beta} - \sqrt{2\alpha}, \quad \sqrt{-2\hat{\delta}} = \sqrt{-2\delta} \neq 0, \quad (A.4)$$

where  $y = y(t)$  and  $\hat{y} = \hat{y}(t)$  solve  $P_5$  for the parameters  $\alpha, \beta, \gamma$ , and  $\delta$  and  $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ , and  $\hat{\delta}$ , respectively. The important feature of this transformation is that the branches of the square roots in equations (A.1)–(A.4) can be taken arbitrarily but their choices remain the same in all the formulae.

Our goal here is to discuss the spectral interpretation of this transformation. For this purpose we use the Jimbo–Miwa [20] isomonodromy representation of  $P_5$ , the Fuchs–Garnier pair in the Jimbo–Miwa parametrization. Consider the following linear matrix ODE:

$$\frac{d\Psi}{d\lambda} = \left( \frac{t}{2}\sigma_3 + \frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} \right) \Psi. \quad (A.5)$$

Here  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  and the matrices  $A_p$  ( $p = 0, 1$ ) are independent of  $\lambda$ . Assume the following parametrization of the matrices  $A_p$ ,

$$A_0 = \begin{pmatrix} z + \frac{\theta_0}{2} & -u(z + \theta_0) \\ z/u & -z - \frac{\theta_0}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} -z - \frac{\theta_0 + \theta_\infty}{2} & uy(z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}) \\ -\frac{1}{uy}(z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}) & z + \frac{\theta_0 + \theta_\infty}{2} \end{pmatrix}. \quad (A.6)$$

Then, the isomonodromy deformations of equation (A.5) with respect to  $t$  are governed by the following system of nonlinear ODEs, which we will call the Isomonodromy Deformation System (IDS)

$$t \frac{dy}{dt} = ty - 2z(y-1)^2 - (y-1) \left( \frac{\theta_0 - \theta_1 + \theta_\infty}{2} y - \frac{3\theta_0 + \theta_1 + \theta_\infty}{2} \right), \quad (\text{A.7})$$

$$t \frac{dz}{dt} = yz \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) - \frac{1}{y} (z + \theta_0) \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right), \quad (\text{A.8})$$

$$t \frac{d}{dt} \log u = -2z - \theta_0 + y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right) + \frac{1}{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right). \quad (\text{A.9})$$

In this system  $\theta_\nu$  ( $\nu = 0, 1, \infty$ ) are complex constants considered as parameters. Following [20] we call them the formal monodromies. Another widely used terminology for  $\theta_\nu$  is the exponential differences, since they coincide with the differences of the eigenvalues of the matrices  $A_\nu$ <sup>9</sup>.

Excluding the function  $z$  from equations (A.7)–(A.8) one finds that the function  $y$  satisfies the fifth Painlevé equation (1.3) for the set of the coefficients (4.22). In this case there exists a fundamental solution of equation (A.5) which solves the following equation:

$$\frac{d\Psi}{dt} = \left( \frac{\lambda}{2} \sigma_3 + \frac{1}{t} (A_0 + A_1 + \frac{\theta_\infty}{2} \sigma_3) \right) \Psi. \quad (\text{A.10})$$

We note that by rescaling  $t$  we may set the coefficients  $\hat{\delta} = \delta = -1/2$ , and hence we may further put

$$\sqrt{-2\delta} = \sqrt{-2\hat{\delta}} = \varepsilon = \pm 1$$

in equations (A.1)–(A.4). To take into account the possibility of different choices of branches of the square roots in equations (A.1)–(A.4) we introduce the parameters  $\varepsilon_1, \varepsilon_2, \hat{\varepsilon}_1, \hat{\varepsilon}_2$ , each taking the value  $\pm 1$ , in the following way

$$\sqrt{2\alpha} = \varepsilon_1 \frac{\theta_0 - \theta_1 + \theta_\infty}{2}, \quad \sqrt{-2\beta} = \varepsilon_2 \frac{\theta_0 - \theta_1 - \theta_\infty}{2}, \quad \gamma = 1 - \theta_0 - \theta_1, \quad (\text{A.11})$$

$$\sqrt{2\hat{\alpha}} = \hat{\varepsilon}_1 \frac{\hat{\theta}_0 - \hat{\theta}_1 + \hat{\theta}_\infty}{2}, \quad \sqrt{-2\hat{\beta}} = \hat{\varepsilon}_2 \frac{\hat{\theta}_0 - \hat{\theta}_1 - \hat{\theta}_\infty}{2}, \quad \hat{\gamma} = 1 - \hat{\theta}_0 - \hat{\theta}_1. \quad (\text{A.12})$$

By substituting equations (A.11) and (A.12) into formulae (A.2)–(A.4) we get the following equations relating the formal monodromies:

$$\hat{\theta}_\infty = \varepsilon \frac{\hat{\varepsilon}_1 - \hat{\varepsilon}_2}{2} (1 - \theta_0 - \theta_1) + \frac{\hat{\varepsilon}_1 + \hat{\varepsilon}_2}{2} \left( 1 - \frac{\varepsilon_1 + \varepsilon_2}{2} (\theta_0 - \theta_1) - \frac{\varepsilon_1 - \varepsilon_2}{2} \theta_\infty \right), \quad (\text{A.13})$$

$$\hat{\theta}_0 - \hat{\theta}_1 = \varepsilon \frac{\hat{\varepsilon}_1 + \hat{\varepsilon}_2}{2} (1 - \theta_0 - \theta_1) + \frac{\hat{\varepsilon}_1 - \hat{\varepsilon}_2}{2} \left( 1 - \frac{\varepsilon_1 + \varepsilon_2}{2} (\theta_0 - \theta_1) - \frac{\varepsilon_1 - \varepsilon_2}{2} \theta_\infty \right), \quad (\text{A.14})$$

$$\hat{\theta}_0 + \hat{\theta}_1 = 1 + \varepsilon \left( \frac{\varepsilon_1 - \varepsilon_2}{2} (\theta_0 - \theta_1) + \frac{\varepsilon_1 + \varepsilon_2}{2} \theta_\infty \right). \quad (\text{A.15})$$

Equations (A.13)–(A.15) define  $2^5 = 32$  different relations for the formal monodromies  $\hat{\theta}$ 's, according to the number of tuples  $(\varepsilon, \varepsilon_1, \varepsilon_2, \hat{\varepsilon}_1, \hat{\varepsilon}_2)$ . It is easy to notice that all these formulae can be presented as the compositions of the actions on the  $\theta$ -parameters of the Schlesinger transformations “dressing” the points at infinity and zero points:

$$S_{\pm, \pm} : \quad \theta_\infty \rightarrow \theta_\infty \pm 1, \quad \theta_0 \rightarrow \theta_0 \pm 1, \quad \theta_1 \rightarrow \theta_1, \quad (\text{A.16})$$

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<sup>9</sup> $A_\infty \equiv \text{diag}(A_0 + A_1) = -\theta_\infty \sigma_3 / 2$

with possibly the reflections:

$$R_0 : \quad \theta_0 \rightarrow -\theta_0, \quad \theta_1 \rightarrow \theta_1, \quad \theta_\infty \rightarrow \theta_\infty, \quad (\text{A.17})$$

$$R_1 : \quad \theta_0 \rightarrow \theta_0, \quad \theta_1 \rightarrow -\theta_1, \quad \theta_\infty \rightarrow \theta_\infty, \quad (\text{A.18})$$

$$R_\infty : \quad \theta_0 \rightarrow \theta_0, \quad \theta_1 \rightarrow \theta_1, \quad \theta_\infty \rightarrow -\theta_\infty, \quad (\text{A.19})$$

$$R_{01} : \quad \theta_0 \rightarrow \theta_1, \quad \theta_1 \rightarrow \theta_0, \quad \theta_\infty \rightarrow \theta_\infty, \quad (\text{A.20})$$

and the following Okamoto-like transformation, mixing the  $\theta$ -variables:

$$\mathcal{O} : \quad \hat{\theta}_0 = \frac{\theta_0 + \theta_1 - \theta_\infty}{2}, \quad \hat{\theta}_1 = \frac{\theta_0 + \theta_1 + \theta_\infty}{2}, \quad \hat{\theta}_\infty = \theta_1 - \theta_0. \quad (\text{A.21})$$

We call the last (mixing  $\theta$ 's) transformation the Okamoto transformation because it coincides with the reflection  $s_3 = \mathcal{O}$ , introduced in Okamoto's work [36]. In general, we call the Okamoto-like transformation any transformation for the Painlevé equations that acts on the space of the corresponding formal monodromies as a linear operator with the matrix having at least one row with nonzero elements: this condition means that it “mixes” all formal monodromies. Note that the absolute value of determinants of the matrices defining these linear operators is always unity. These transformations appeared in Okamoto's studies of the Painlevé equations “on a regular footing” [35]–[37]. They represent one of the reflections in the subgroup of affine Weyl symmetries and exist for  $P_4$ ,  $P_5$ , and  $P_6$ .

It is important to mention that the set of transformations (A.16)–(A.21) is slightly wider than those transformations that can be obtained via making compositions of Bäcklund transformation (A.1) with different choices of the branches, e.g., from the latter transformation one can obtain only the reflection  $R_\infty \circ R_{01}$  rather than two of them separately. As we see below, to produce  $R_\infty$  and  $R_{01}$  we need an additional reflection of  $t \rightarrow -t$ , so that it is not a transformation of the first kind in Okamoto's sense [36].

It is clear that among these transformations only 4 are independent, say,  $S_{+,-}$ ,  $R_0$ ,  $R_\infty$ ,  $\mathcal{O}$ , the others can be presented as follows:  $R_{01} = R_\infty \circ (R_\infty \circ \mathcal{O})^2$ ,  $R_1 = R_{01} \circ R_0 \circ R_{01}$ ,  $S_{+,+} = R_{01} \circ S_{+,-} \circ R_{01}$ , etc.

Now we are ready to discuss the presentation of these transformations on the solutions of the Fuchs-Garnier pair (A.5), (A.10).

It is well known that the Schlesinger transformations  $S_{\pm,\pm}$ , are the gauge transformation of the  $\Psi$  function,  $\Psi \rightarrow S(\lambda, t)/\sqrt{\lambda}\Psi$ , where  $S(\lambda, t)$  is a linear function of  $\lambda$ . The general theory for these transformations in the framework of the isomonodromy deformations can be found in [20] and particular formulae for  $P_5$  in [31].

Reflections  $R_0$  and  $R_1$  do not have any spectral representation. For each  $k = 0, 1$  both numbers:  $+\theta_k/2$  and  $-\theta_k/2$ , are the eigenvalues of the corresponding residue matrix  $A_k$  and because  $\Psi$  is not normalized in the neighborhood of the singular points  $\lambda = 0$  and  $\lambda = 1$ , it does not “feel a difference” between  $+\theta_k$  and  $-\theta_k$   $k = 0, 1$ . The last statement means that for each choice of the sign of  $\theta_k$ 's in the neighborhood of each singular point,  $\lambda = 0$  and  $\lambda = 1$ , there exists its own series expansion representing the solution. However, this fact does not affect the corresponding monodromy matrices or the residues. For example, consider  $R_1$ . Denote, with hats the Painlevé functions after this reflection (see equations (A.5) and (A.6)):

$$\begin{aligned} R_1 : \quad & \hat{\theta}_0 = \theta_0, \quad \hat{\theta}_1 = -\theta_1, \quad \hat{\theta}_\infty = \theta_\infty \\ \text{from } A_0 : \quad & \hat{z} = z, \quad \hat{u} = u, \\ \text{from } A_1 : \quad & \hat{y} \left( z + \frac{\theta_0 + \theta_1 + \theta_\infty}{2} \right) = y \left( z + \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \right). \end{aligned} \quad (\text{A.22})$$

The last equation represents the Bäcklund transformation of  $P_5$  corresponding to the reflection  $R_1$ . Substituting  $z$  from equation (A.7) one finds quite complicated explicit

formula for  $\hat{y}$  in terms of  $y$ . In an analogous way we find the nonlinear action of  $R_0$ :

$$\begin{aligned} R_0 : \quad & \hat{\theta}_0 = -\theta_0, \quad \hat{\theta}_1 = \theta_1, \quad \hat{\theta}_\infty = \theta_\infty \\ \text{from } A_0 : \quad & \hat{z} - \frac{\theta_0}{2} = z + \frac{\theta_0}{2}, \quad \frac{\hat{z}}{\hat{u}} = \frac{z}{u}, \\ \text{from } A_1 : \quad & \hat{u}\hat{y} = uy \Rightarrow \hat{y} = y \frac{z}{z + \theta_0}, \quad \hat{z} = z + \theta_0, \quad \hat{u} = u \frac{z + \theta_0}{z}. \end{aligned} \quad (\text{A.23})$$

Now we have two points of view: the Jimbo-Miwa parametrization is very good since it allows one to obtain very easily quite nontrivial Bäcklund transformations for  $P_5$ ; on the other hand, there is a discrepancy between linear and nonlinear actions of the transformations  $R_0$  and  $R_1$ , no linear action “produces” a nontrivial nonlinear action. Our result in Subsection 4.5 related with the action of the Okamoto transformation  $\mathcal{O}$  means it is possible to obtain another parametrization of Fuchs-Garnier pair (A.5), (A.10), where  $R_0$  and  $R_1$  do not produce any nonlinear action on  $y$ , see Remark 4.2.

The situation with the reflection  $R_\infty$  is different because of the normalization of the  $\Psi$  function at  $\lambda = \infty$ : the fact that at this point  $\Psi$  has an irregular singularity does not play an essential role in this question. The linear action reads

$$\hat{\Psi}(\lambda, \hat{t}) = \sigma_1 \Psi(\lambda, t) \sigma_1, \quad \hat{t} = -t, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The corresponding Bäcklund transformation is

$$\hat{y}(\hat{t}) = 1/y(t), \quad \hat{z}(\hat{t}) = -z(t) - \theta_0, \quad \hat{u}(\hat{t}) = 1/u(t).$$

The linear representation for  $R_{01}$  is as follows:

$$\hat{\Psi}(\hat{\lambda}, \hat{t}) = e^{-t\sigma_3/2} \Psi(\lambda, t), \quad \hat{\lambda} = 1 - \lambda, \quad \hat{t} = -t.$$

It generates the following nonlinear representation,

$$\hat{y}(\hat{t}) = 1/y(t), \quad \hat{z}(\hat{t}) = -z(t) - \frac{\theta_0 + \theta_1 + \theta_\infty}{2}, \quad \hat{u}(\hat{t}) = e^{-t} y(t) u(t).$$

The transformation that we obtain in Subsection 4.5, equation (4.53), coincides with the composition  $R_1 \circ \mathcal{O}$ . As is explained above,  $R_1$ , being a nontrivial transformation for  $P_5$ , is not observable from “spectral point of view”, so the same procedure gives us exactly  $\mathcal{O}$  just by choosing a different parametrization of the Fuchs-Garnier pair. The linear representation of the Okamoto transformation is given in (4.56).

Let us present the “true” Jimbo-Miwa parametrization of the Fuchs-Garnier pair (A.5), (A.10). It is obtained from the original Jimbo-Miwa parametrization (A.6) by substituting in it instead of  $y$  and  $z$  their expressions in terms of  $\hat{y}$  and  $\hat{z}$ , obtained from the first two equations in (4.53), a redefinition of  $u = \hat{u}(\hat{z} - (\theta_0 - \theta_1 + \theta_\infty)/2)$ , the shift  $\hat{z} + \theta_1/2 \rightarrow \hat{z}$ , and removing the hats:

$$A_0 = \begin{pmatrix} z - \frac{\theta_\infty}{2} & -u \left( \left( z - \frac{\theta_\infty}{2} \right)^2 - \frac{\theta_0^2}{4} \right) \\ \frac{1}{u} & -z + \frac{\theta_\infty}{2} \end{pmatrix}, \quad A_1 = \begin{pmatrix} -z & uy \left( z^2 - \frac{\theta_1^2}{4} \right) \\ -\frac{1}{uy} & z \end{pmatrix}. \quad (\text{A.24})$$

The system of isomonodromy deformations in this parametrization reads:

$$t \frac{dy}{dt} = ty - 2z(y-1)^2 - \theta_\infty(y-1), \quad (\text{A.25})$$

$$t \frac{dz}{dt} = y \left( z^2 - \frac{\theta_1^2}{4} \right) - \frac{1}{y} \left( \left( z - \frac{\theta_\infty}{2} \right)^2 - \frac{\theta_0^2}{4} \right), \quad (\text{A.26})$$

$$t \frac{d}{dt} \log u = 2 \left( z - \frac{\theta_\infty}{2} \right) \left( \frac{1}{y} - 1 \right). \quad (\text{A.27})$$

Eliminating  $z$  from equation (A.25) and substituting it into equation (A.26) one finds that  $y$  solves  $P_5$  (1.3) for the parameters:

$$\alpha = \frac{\theta_1^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = 1 + \theta_\infty, \quad \delta = -\frac{1}{2}. \quad (\text{A.28})$$

Compare equations (A.25)–(A.27) and (A.28) with the original ones by Jimbo–Miwa (A.7)–(A.9) and (4.22).

Note, that now formulae (A.28) look similar to the analogous formulae for coefficients of the other Painlevé equations in the Jimbo–Miwa parameterizations. To make this parametrization absolutely perfect we can apply transformation  $R_{01}$ , to get  $P_5$  with  $\alpha = \theta_0^2/2$  and  $\beta = -\theta_1^2/2$ .

We also remark that in this parametrization, changing of the signs of  $\theta_0$  and  $\theta_1$  has no effect on  $y$  and different transformations of  $\Psi$  correspond to different transformations of  $y$ . This was not the case for the original Jimbo–Miwa parametrization.

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